Arătaţi că pentru orice număr natural nenul n are loc inegalitatea

$$\frac{1^2+1-1}{2!}+\frac{2^2+2-1}{3!}+\frac{3^2+3-1}{4!}+\cdots+\frac{n^2+n-1}{(n+1)!}<2,$$

unde, pentru $k\in\mathbb{N}^*$, $k!=1\cdot 2\cdot 3\cdot\ldots\cdot k$.

* * *

Soluţie: Definind $0!=1$, avem pentru orice $k\in\mathbb{N}$, $k\geq 1$

$$\frac{k^2+k-1}{(k+1)!}=\frac{k(k+1)-1}{(k+1)!}=\frac{k(k+1)}{(k+1)!}-\frac{1}{(k+1)!}=\frac{1}{(k-1)!}-\frac{1}{(k+1)!}, \qquad (*)$$

deoarece $(k+1)!=\underbrace{1\cdot 2\cdot 3\cdot\ldots(k-1)}_{=(k-1)!}\cdot k\cdot(k+1)=(k-1)!\cdot k(k+1)$.

Folosind relaţia $(*)$ pentru fiecare $k=1,2,\ldots n$, obţinem că

$$\frac{1^2+1-1}{2!}+\frac{2^2+2-1}{3!}+\frac{3^2+3-1}{4!}+\cdots+\frac{n^2+n-1}{(n+1)!}=$$

$$=\left(\frac{1}{0!}-\frac{1}{2!}\right)+\left(\frac{1}{1!}-\frac{1}{3!}\right)+\left(\frac{1}{2!}-\frac{1}{4!}\right)+\left(\frac{1}{3!}-\frac{1}{5!}\right)+\left(\frac{1}{4!}-\frac{1}{6!}\right)+\cdots+$$

$$=\left(\frac{1}{(n-4)!}-\frac{1}{(n-2)!}\right)+\left(\frac{1}{(n-3)!}-\frac{1}{(n-1)!}\right)+\left(\frac{1}{(n-2)!}-\frac{1}{n!}\right)+$$

$$+\left(\frac{1}{(n-1)!}-\frac{1}{(n+1)!}\right)=\frac{1}{0!}+\frac{1}{1!}-\frac{1}{n!}-\frac{1}{(n+1)!}=2-\frac{1}{n!}-\frac{1}{(n+1)!}<2.$$