The 5th "STARS of MATHEMATICS" Competition – Juniors
December 10, 2011 ★★★ ICHB – Bucharest

Solutions

Problem 1. For positive real numbers a,b,c,d, with $abcd=1$, determine all values taken by the expression

$$\frac{1+a+ab}{1+a+ab+abc}+\frac{1+b+bc}{1+b+bc+bcd}+\frac{1+c+cd}{1+c+cd+cda}+\frac{1+d+da}{1+d+da+dab}.$$

DAN SCHWARZ (BASED ON A SWISS OLYMPIAD)

Solution. We have

$$\begin{aligned} a(1+b+bc+bcd) &= 1+a+ab+abc,\\ b(1+c+cd+cda) &= 1+b+bc+bcd,\\ c(1+d+da+dab) &= 1+c+cd+cda. \end{aligned}$$

Then by amplifying the second, third and fourth terms by a, ab, respectively abc, we get a fraction with denominator $1+a+ab+abc$, and numerator

$$(1+a+ab)+a(1+b+bc)+ab(1+c+cd)+abc(1+d+da)$$

equalling $3(1+a+ab+abc)$, therefore the expression has a constant value, equal to $\boxed{3}$. ■

Alternative Solution. Using a well-known method of substitution, take $a=\frac{x}{y}$, $b=\frac{y}{z}$, $c=\frac{z}{t}$, $d=\frac{t}{x}$. Then $1+a+ab+abc=\frac{x}{x}+\frac{x}{y}+\frac{x}{z}+\frac{x}{t}=x\frac{xyz+yzt+ztx+txy}{xyzt}$, while $1+a+ab=\frac{x}{x}+\frac{x}{y}+\frac{x}{z}=x\frac{xy+yz+zx}{xyz}$.

So $\frac{1+a+ab}{1+a+ab+abc}=\frac{t(xy+yz+zx)}{xyz+yzt+ztx+txy}$. Performing the computations on the other three terms, and summing up, we get for the expression the constant value 3. ■

Remark. One can similarly obtain the following identities

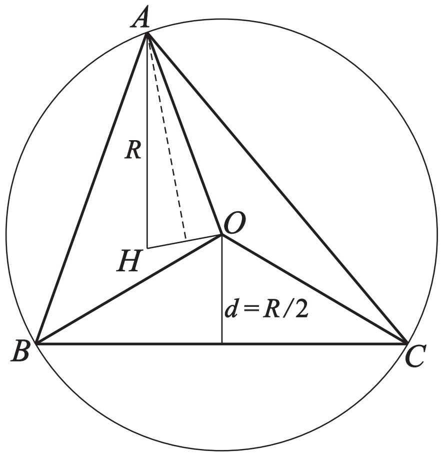
$$\sum_{cyc}\frac{1+a}{1+a+ab+abc}=2 \text{ and } \sum_{cyc}\frac{1}{1+a+ab+abc}=1.$$

The situation easily generalizes to $n\geq 3$ positive variables $x_1,x_2,\ldots,x_n$ with unit product. Then, for any $1\leq k\leq n$, one can establish the identity (where by convention $\prod_{i=1}^{0}x_i=1$)

$$\sum_{cyc}\left(\left(\sum_{j=0}^{k-1}\prod_{i=1}^{j}x_i\right)\Big/\left(\sum_{j=0}^{n-1}\prod_{i=1}^{j}x_i\right)\right)=k.$$

Problem 2. Let ABC be an acute-angled, not equilateral triangle, where vertex A lies on the perpendicular bisector of the segment HO, joining the orthocentre H to the circumcentre O. Determine all possible values for the measure of angle A.

IMO 1989 SHORTLIST - U.S.A.



Solution. Since $\triangle ABC$ is acute-angled, both H and O are interior to it. We have $AH=AO=R$ (the circumradius). By a well-known result, AH is twice the distance d from O to the side BC, so $d=R/2$. Then $\angle OBC=\angle OCB=30°$. But $\angle OBA=\angle OAB$ and $\angle OCA=\angle OAC$ (in isosceles triangles).

It follows that $180°=\angle A+\angle B+\angle C=(\angle OAB+\angle OAC)+(\angle OBA+\angle OBC)+(\angle OCA+\angle OCB)=2\angle A+60°$, whence $\boxed{\angle A=60°}$. ■

Alternative Solution. (Vectorial) Let the circumcentre O be the origin of a vector system in the plane. Consequently, $|\vec{A}|=|\vec{B}|=|\vec{C}|=R$ (the circumradius), and $\vec{H}=\vec{A}+\vec{B}+\vec{C}$, by Sylvester's relation. We are given that $R=|\vec{H}-\vec{A}|$.

We can now compute (via dot-product manipulations) $R^2=\langle\vec{H}-\vec{A},\vec{H}-\vec{A}\rangle=\langle\vec{B}+\vec{C},\vec{B}+\vec{C}\rangle=2\langle\vec{B},\vec{B}\rangle+2\langle\vec{C},\vec{C}\rangle-\langle\vec{B}-\vec{C},\vec{B}-\vec{C}\rangle=2|\vec{B}|^2+2|\vec{C}|^2-|\vec{B}-\vec{C}|^2=4R^2-|\vec{B}-\vec{C}|^2$.

It follows that $\sin A=|\vec{B}-\vec{C}|/R=\sqrt{3}/2$, hence $\angle A=60°$.

Equivalently, $R^2=\langle\vec{B}+\vec{C},\vec{B}+\vec{C}\rangle=|\vec{B}|^2+|\vec{C}|^2+2\langle\vec{B},\vec{C}\rangle=2R^2+2R^2\cos 2A$, hence $2\angle A=120°$. ■

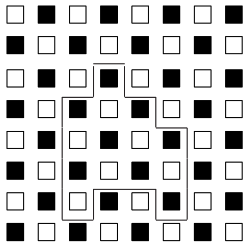
Alternative Solution. (Trigonometric) Let now CC' be the altitude from C (with C' lying on the side AB). Since $AH=R$ (the circumradius), we have $AC'=R\sin B$, and so $CC'=R\sin B\tan A$. From the relations $\sin A=BC/2R$ and $CC'=BC\sin B$, we have $CC'=2R\sin A\sin B$; in fact this proves $\boxed{AH=2R\cos A}$. Putting it all together yields $\cos A=1/2$, hence $\angle A=60°$. ■

Remark. Notice that the converse is also true; in an acute-angled triangle with $\angle A=60°$, the vertex A is shown to be equidistant from H and O, since then the distance d from O to BC is $R/2$, and so $AH=2d=R=AO$.

Dinu Şerbănescu notices that by removing the condition that $\triangle ABC$ be acute-angled, a second possibility arises, namely $\angle A=120°$.

Problem 3. The checkered plane is painted black and white, after a chessboard fashion. A polygon Π of area S and perimeter P consists of some of these unit squares (i.e., its sides go along the borders of the squares). Prove polygon Π contains not more than $\frac{S}{2}+\frac{P}{8}$, and not less than $\frac{S}{2}-\frac{P}{8}$ squares of a same color.

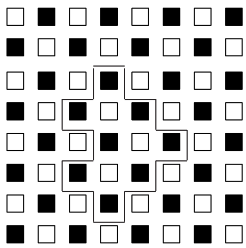
MOFM 2011 SHORTLIST - ALEXANDER MAGAZINOV



A polygon with $S = 14, P = 20$ and $8 < \frac{S}{2} + \frac{P}{8}$ black squares.

Solution. Consider all b black squares inside Π; they have area b, and the number of their sides is $4b$. At least $4b - P$ of these sides do not belong to the boundary of Π and therefore lie on the border of some white squares inside Π. This means there are at least $\frac{4b-P}{4}$ white squares inside Π, and $S \geq b + \frac{4b-P}{4} = 2b - \frac{P}{4}$, which is equivalent to $\boxed{b \leq \frac{S}{2} + \frac{P}{8}}$ as desired. The same proof applies to the white squares, and since the total number of squares is clearly equal to S, the other inequality $\boxed{b \geq \frac{S}{2} - \frac{P}{8}}$ follows. A same inequality holds for the white squares.

Notice that the estimate is sharp for those polygons, and only those, whose boundary is monochromatic. ■



For this polygon $S = 11$, $P = 20$, and $b = 8 = \frac{S}{2} + \frac{P}{8}$.

Alternative Solution. (Dan Schwarz) Generalization by induction on S. Consider, instead of a polygon, any finite collection of squares, for which we define the *perimeter* as being the total number of sides, minus those shared by two squares (so the definition remains consistent with that for a polygon). Now, the removal of a white square will decrease S by 1, and can only increase the perimeter by 4, so by the induction hypothesis $b \leq \frac{S-1}{2} + \frac{P+4}{8} = \frac{S}{2} + \frac{P}{8}$. After all white square have thus been removed, we are only left with black squares $b = \frac{b}{2} + \frac{4b}{8} = \frac{S_b}{2} + \frac{P_b}{8}$ (where S_b and P_b are the area, respectively perimeter of the black collection).

Now that the wanted inequality is established for the black squares, the complementary one follows from the similar bound on the white squares, as seen in the previous solution. ■

Problem 4. Let $n \geq 2$ be an integer. Let us call *interval* a subset $A \subseteq \{1,2,\dots,n\}$ for which integers $1 \leq a < b \leq n$ do exist, such that $A = \{a, a+1, \dots, b-1, b\}$. Let a family $\mathscr{A}$ of subsets $A_i \subseteq \{1,2,\dots,n\}$, with $1 \leq i \leq N$, be such that for any $1 \leq i < j \leq N$ we have $A_i \cap A_j$ being an interval. Prove that $N \leq \lfloor n^2/4 \rfloor$, and that this bound is sharp.

Dan Schwarz (after an idea by Ron Graham)

Solution. Let $A_i \subseteq B_i = [\min A_i, \max A_i] \cap \{1,2,\dots,n\}$ for $1 \leq i \leq N$, thus B_i are intervals. Then if $B_i = B_j$ for some $i \neq j$, we also have $A_i = A_j$, since $A_i \cap A_j$ is an interval. From $A_i \cap A_j \subseteq B_i \cap B_j$ follows that $|B_i \cap B_j| \geq |A_i \cap A_j| \geq 2$, so the family $\mathscr{B}$ of the subsets B_i also has the mentioned property.

But $\bigcap_{i=1}^{N} B_i$ contains an interval of length (at least) 1; let that be $[a+1, a+2]$. Then there are at most

$$(a+1)(n-a-1) = \left(\frac{n}{2}\right)^2 - \left(a - \frac{n-2}{2}\right)^2$$

possibilities, so $N \leq \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor = \boxed{\lfloor n^2/4 \rfloor}$.

An example of such a maximal family is thus the family of all intervals containing the interval $[\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1]$, and so the bound is sharp. ■

Alternative Solution. (Dan Schwarz) Build the graph $G = (V, E)$, with $V = \{1,2,\dots,n\}$ and $E = \{\{\min A, \max A\} \mid A \in \mathscr{A}\}$ (notice $\{\min A, \max A\} = \{\min A', \max A'\}$ implies $A = A'$, since $A \cap A'$ is an interval for $A \neq A'$), so $|E| = |\mathscr{A}| = N$. But G can contain no triangle $\{1 \leq i < j < k \leq n\}$, since then there will exist two members of $\mathscr{A}$ intersecting in one point only. Thus, by Mantel's theorem (in fact a special case of Turán's theorem), $N \leq \lfloor n^2/4 \rfloor$ and the only model is a quasi-balanced complete bipartite graph, with its shores being $\{\min A \mid A \in \mathscr{A}\}$ and $\{\max A \mid A \in \mathscr{A}\}$, this being such that $\max_{A \in \mathscr{A}} \min A < \min_{A \in \mathscr{A}} \max A$, corresponding to the construction shown above. ■

Remark. One can push this even further. Read on the following more general result, where the case $k = 1$ has been considered by Ron Graham, and which I have generalized for any $1 \leq k \leq n$, the case $k = 2$ being that of our problem (Graham also considered the case of unrestricted pairwise intersections of a family of N subsets being (possibly empty) intervals, with the sharp bound of $N \leq \binom{n}{2} + n + 1$, achieved by the family of subsets of at most two elements – see for it [R. Graham, M. Simonovits and V. Sós – A Note on the Intersection Properties of Subsets of Integers, Journal of Combinatorial Theory, Series A, 1980]).

Problem. Let $1 \leq k \leq n$ be fixed integers. Call *interval* a subset $A \subseteq \{1,2,\dots,n\}$ for which integers $1 \leq a \leq b \leq n$ do exist, such that $A = \{a, a+1, \dots, b\}$. Let a family $\mathscr{A}$ of subsets $A_i \subseteq \{1,2,\dots,n\}$, $1 \leq i \leq N$, be such that for any $1 \leq i < j \leq N$ we have $|A_i \cap A_j| \geq k$, with $A_i \cap A_j$ being an interval. Prove that $N \leq \left\lfloor \left(\frac{n-k+2}{2}\right)^2 \right\rfloor$.

The **Solution** is almost *verbatim* the first one used above.

An example of such a maximal family is then the family of all intervals containing the interval $\left[\left\lfloor \frac{n-k}{2} \right\rfloor + 1, \left\lfloor \frac{n-k}{2} \right\rfloor + k\right]$, and so the bound is sharp. ■