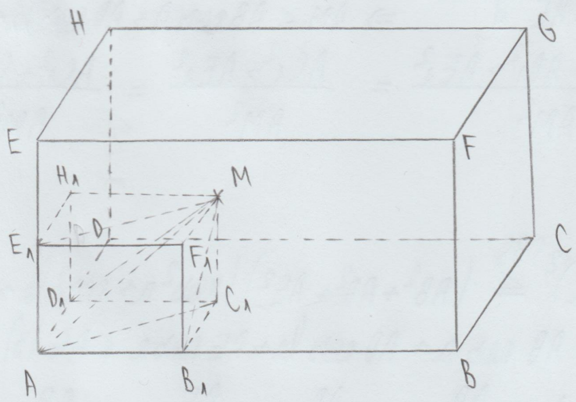
Etapa 6, Problema 3

Fie paralelipipedul dreptunghic $ABCDEFGH$ și M un punct interior. Fie a, b, c măsurile unghiurilor formate de AM cu AB, AD, AE. Demonstrați că

$$AM < AB\cos a + AD\cos b + AE\cos c \leq AG.$$ (enunț corectat)

Soluție.



Construim $MC_1 \perp (ABC)$, $C_1 \in (ABC)$, $C_1B_1 \perp AB$, $B_1 \in (AB)$, $C_1D_1 \perp AD$, $D_1 \in (AD)$, $MF_1 \perp (ABF)$, $F_1 \in (ABF)$, $F_1E_1 \perp AE$, $E_1 \in (AE)$ și $MH_1 \perp (ADH)$, $H_1 \in (ADH)$. $\Rightarrow AB_1C_1D_1E_1F_1MH_1$ paralelipiped dreptunghic

$$\left.\begin{array}{l} AB_1 \perp B_1C_1 \\ AB_1 \perp B_1F_1 \\ B_1C_1, B_1F_1 \subset (B_1C_1M) \end{array}\right\} \Rightarrow \left.\begin{array}{l} AB_1 \perp (B_1C_1M) \\ \text{Dar, } B_1M \subset (B_1C_1M) \end{array}\right\} \Rightarrow AB_1 \perp B_1M$$

$$\left.\begin{array}{l} DD_1 \perp D_1C_1 \\ DD_1 \perp H_1D_1 \\ D_1C_1, H_1D_1 \subset (C_1D_1M) \end{array}\right\} \Rightarrow \left.\begin{array}{l} DD_1 \perp (C_1D_1M) \\ \text{Dar, } D_1M \subset (C_1D_1M) \end{array}\right\} \Rightarrow DD_1 \perp D_1M \Rightarrow AD_1 \perp D_1M$$

$$\left.\begin{array}{l} AE_1 \perp E_1F_1 \\ AE_1 \perp E_1H_1 \\ E_1F_1, E_1H_1 \subset (E_1F_1M) \end{array}\right\} \Rightarrow \left.\begin{array}{l} AE_1 \perp (E_1F_1M) \\ \text{Dar, } E_1M \subset (E_1F_1M) \end{array}\right\} \Rightarrow AE_1 \perp E_1M$$

ΔMB_1A: $m(\widehat{MB_1A}) = 90^\circ$

$$\cos \widehat{MAB_1} = \frac{AB_1}{AM} \Rightarrow AB_1 = AM\cos \widehat{MAB_1} = AM\cos a$$

ΔMD_1A: $m(\widehat{MD_1A}) = 90^\circ$

$$\cos \widehat{MAD_1} = \frac{AD_1}{AM} \Rightarrow AD_1 = AM\cos \widehat{MAD_1} = AM\cos b$$

$\Delta ME_1A: m(\widehat{ME_1A}) = 90^\circ$

$\cos \widehat{MAE_1} = \dfrac{AE_1}{AM} \Rightarrow AE_1 = AM \cos \widehat{MAE_1} = AM \cos c$

$AM^2 = AC_1^2 + C_1M^2 = AB_1^2 + AD_1^2 + AE_1^2 = AB_1 \cdot AB_1 + AD_1 \cdot AD_1 + AE_1 \cdot AE_1 = AM \cos a \, AB_1 +$
$+ AM \cos b \, AD_1 + AM \cos c \, AE_1$

Dar, $AB_1 < AB$, $AD_1 < AD$, $AE_1 < AE$ $\Bigg\} \Rightarrow AM^2 < \cancel{AM}(AB \cos a + AD \cos b + AE \cos c) \Rightarrow$
$\Rightarrow AM < AB \cos a + AD \cos b + AE \cos c$ (1)

$\cos^2 a + \cos^2 b + \cos^2 c = \dfrac{AB_1^2 + AD_1^2 + AE_1^2}{AM^2} = \dfrac{AC_1^2 + AE_1^2}{AM^2} = \dfrac{AC_1^2 + C_1M^2}{AM^2} =$

$= \dfrac{AM^2}{AM^2} = 1$ (2)

$(AB \cos a + AD \cos b + AE \cos c)^2 \overset{CBS}{\leq} (AB^2 + AD^2 + AE^2)(\cos^2 a + \cos^2 b + \cos^2 c) \overset{(2)}{=}$
$= AB^2 + AD^2 + AE^2 = AC^2 + CG^2 = AG^2 \Rightarrow AB \cos a + AD \cos b + AE \cos c \leq AG$ (3)

Egalitatea are loc atunci când $\dfrac{AB}{\cos a} = \dfrac{AD}{\cos b} = \dfrac{AE}{\cos c} \Rightarrow \dfrac{AB}{AB_1} = \dfrac{AD}{AD_1} =$
$= \dfrac{AE}{AE_1}$, ceea ce este posibil atunci când $M \in (AG)$.

Din (1) și (3) $\Rightarrow AM < AB \cos a + AD \cos b + AE \cos c \leq AG$.

Elev Bonifate Tudor-Nicolae
Clasa a X-a
Colegiul Național „I. L. Caragiale" Ploiești
Profesor: Apostolescu Cezar