# The 7th "STARS of MATHEMATICS" Competition – Seniors
## October 19, 2013 ★★★ ICHB – Bucharest

## Solutions

**Problem 1.** Let $\mathscr{F}$ be the family of bijective increasing functions $f\colon[0,1]\to[0,1]$, and let $a\in(0,1)$. Determine the best constants $m_a$ and $M_a$, such that for all $f\in\mathscr{F}$ we have

$$m_a \le f(a)+f^{-1}(a)\le M_a.$$

DAN SCHWARZ

**Solution.** Clearly, for $f\in\mathscr{F}$ we have $f(0)=0$ and $f(1)=1$. We claim $M_a=\sup\{f(a)+f^{-1}(a)\mid f\in\mathscr{F}\}=\boxed{1+a}$ for $a\in(0,1)$, without being reached for any function $f\in\mathscr{F}$ (obviously, of excluded cases, $M_0=0$ and $M_1=2$, reached for any function $f\in\mathscr{F}$).

We may assume $f(a)\le a$, otherwise we work with $f^{-1}$ (since certainly $f^{-1}\in\mathscr{F}$, and from $f(a)\ge a$ would follow $a\ge f^{-1}(a)$). But $f^{-1}(a)<1$, therefore $f(a)+f^{-1}(a)<a+1$. For the other part, let us take $0<\varepsilon<2(1-a)$. Consider the points $O(0,0)$, $A_\varepsilon(a,a-\varepsilon/2)$, $B_\varepsilon(1-\varepsilon/2,a)$, $I(1,1)$. Take $f_\varepsilon$ to be the piecewise linear function having the broken line $OA_\varepsilon B_\varepsilon I$ as its graph. Then, considering the points $A'_\varepsilon(a-\varepsilon/2,a)$, $B'_\varepsilon(a,1-\varepsilon/2)$, the function $f_\varepsilon^{-1}$ will be the piecewise linear function having the broken line $OA'_\varepsilon B'_\varepsilon I$ as graph. Under this construction

$$f_\varepsilon(a)+f_\varepsilon^{-1}(a)=(a-\varepsilon/2)+(1-\varepsilon/2)=(a+1)-\varepsilon.$$

We similarly find $m_a=\inf\{f(a)+f^{-1}(a)\mid f\in\mathscr{F}\}=\boxed{a}$ for $a\in(0,1)$. A more elegant way, once the above result, runs like this.

Define the operator $T\colon\mathscr{F}\to\mathscr{F}$ by $Tf(x)=1-f(1-x)$ (this is well-defined, since clearly $Tf\colon[0,1]\to[0,1]$ is bijective increasing). It is now immediately verified $T$ is involutive, i.e. $T^2f=f$, and also $(Tf)^{-1}=Tf^{-1}$. Therefore $(Tf)(a)+(Tf)^{-1}(a)=Tf(a)+Tf^{-1}(a)$, evaluating as

$$(1-f(1-a))+(1-f^{-1}(1-a))=2-(f(1-a)+f^{-1}(1-a)).$$

But $\{f(a)+f^{-1}(a)\mid f\in\mathscr{F}\}=\{(Tf)(a)+(Tf)^{-1}(a)\mid f\in\mathscr{F}\}$ (since $T$ is bijective), and $m_a=2-M_{1-a}=2-(1+(1-a))=a$.

Thus the answer is $m_a=a$ and $M_a=1+a$, for which

$$\boxed{a<f(a)+f^{-1}(a)<1+a}$$

■

**Problem 2.** Three points inside a rectangle determine a triangle. A fourth point is taken inside the triangle.

**i**) Prove at least one of the three concave quadrilaterals formed by these four points has perimeter lesser than that of the rectangle.

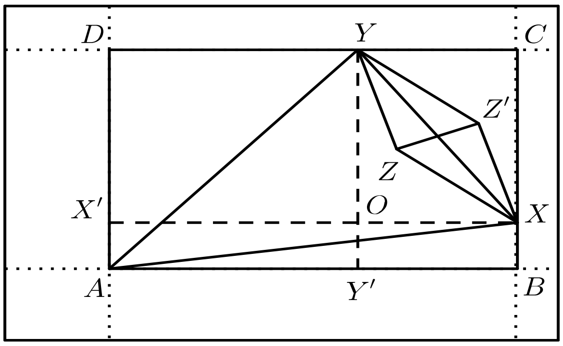
**ii**) Assuming the three points inside the rectangle are three corners of it, prove at least two of the three concave quadrilaterals formed by these four points have perimeters lesser than that of the rectangle.

DAN SCHWARZ

**Solution.**

i) Draw lines, parallel to the sides of the given rectangle, through the vertices of the triangle and consider the least rectangle $ABCD$ thus formed which contains the triangle. In the general case, one vertex of the triangle is situated at $A$, while the other two are, say $X$ on the side $BC$ and $Y$ on the side $DC$ (with the possibility that they are even situated at $B$, $C$ or $D$). The fourth point $Z$ is inside $\triangle AXY$. Denote by $X'$ the point on side $AD$ such that $XX'\parallel AB$ and by $Y'$ the point on side $AB$ such that $YY'\parallel AD$. Also denote by $O$ the meeting point of $XX'$ and $YY'$.

The (some of them possibly degenerated) triangles $AXX'$, $AYY'$ and $XOY$ together cover the triangle $AXY$. Say $Z$ lies in $\triangle XOY$; then its symmetric $Z'$ with respect to the midpoint of $XY$ is contained in the rectangle $ABCD$. But the quadrilateral $AXZ'Y$ is convex, and also contained in the rectangle $ABCD$, thus by a well-known property of plane convex bodies, its perimeter is less than that of $ABCD$, in turn less than that of the original rectangle. Since $ZX=Z'Y$ and $ZY=Z'X$, the perimeter of $AXZY$ is equal to that of $AXZ'Y$, and we are done. The other cases, when $Z$ lies in $\triangle AXX'$ or $\triangle AYY'$ (or both), are treated in a completely similar way. This very question has also been asked in the Juniors' paper.



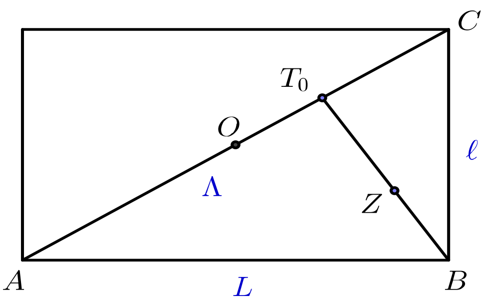

Figures courtesy of ANDREI ECKSTEIN.

ii) Say this triangle is $ABC$, with point $Z$ inside. Then $AZCB$ has lesser perimeter than that of the rectangle, since $AZ+ZC<AB+BC$ (unless $Z\equiv B$, but then the other two are eligible). So we need one more from among $BZCA$ and $BZAC$. Draw line $BZ$ until it meets $AC$ at $T$; also denote by $O$ the midpoint of $AC$, and by $L$, $\ell$, respectively $\Lambda$ the lengths of $AB$, $BC$, respectively $AC$, wlog with $0<\ell\le L$, and of course $L<\Lambda=\sqrt{L^2+\ell^2}<L+\ell$.

Since $BZ+ZC\le BT+TC$ and $BZ+ZA\le BT+TA$, it is enough (and needed, since $Z$ could be at $T$) to prove the claim for concave quadrilaterals $BTCA$ and $BTAC$. Let $P_1(T)=\Lambda+L+TC+BT$ be the perimeter of $BTCA$, and let $P_2(T)=\Lambda+\ell+TA+BT$ be the perimeter of $BTAC$. There exists a unique point $T_0$ on $AC$ such that $P_1(T_0)=P_2(T_0)$, for $T_0$ lying on $OC$ when $OT_0=\dfrac{L-\ell}{2}$. We have $P_1(T)\le P_1(T_0)$ for $T$ lying on $T_0C$ and $P_2(T)\le P_2(T_0)$ for $T$ lying on $AT_0$, so it is enough to prove $P_1(T_0)+P_2(T_0)<2(2L+2\ell)$, writing as

$$2BT_0<3(L+\ell-\Lambda).$$

We have $\cos\dfrac{1}{2}\angle BOC=\dfrac{L}{\Lambda}$, and so $\cos\angle BOC=\dfrac{L^2-\ell^2}{\Lambda^2}$.




By the cosine theorem

$$4BT_0^2 = \Lambda^2 + (L-\ell)^2 - 2\Lambda(L-\ell)\frac{L^2-\ell^2}{\Lambda^2}.$$

On the other hand, $(L+\ell-\Lambda)^2 = 2\Lambda^2 + 2L\ell - 2\Lambda(L+\ell)$ and $(L-\ell)^2 = \Lambda^2 - 2L\ell$, so we can write

$$4BT_0^2 = (L+\ell-\Lambda)^2 + \frac{4L\ell(L+\ell-\Lambda)}{\Lambda}.$$

The (squared) inequality thus rewrites as

$$\frac{L+\ell-\Lambda}{\Lambda}\big(L\ell - 2\Lambda(L+\ell-\Lambda)\big) < 0,$$

but $2\Lambda(L+\ell-\Lambda) = \dfrac{4\Lambda L\ell}{L+\ell+\Lambda} > L\ell$, since being equivalent with $L\ell(3\Lambda - L - \ell) > 0$, obviously true. ■

**Remarks.** The inequalities are strict, but sharp, since they turn into equalities for $\ell \to 0$. It is believed this result holds in general, for all configurations allowed in the statement; alas – the computations seem too hard. Maybe someone can come up with some palatable general proof.

**Problem 3.** Consider the sequence $(a^n+1)_{n\geq 1}$, with $a>1$ a fixed integer.

**i**) Prove there exist infinitely many primes, each dividing some term of the sequence.
**ii**) Prove there exist infinitely many primes, none dividing any term of the sequence.

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**Solution.**
i) In a similar vein with the known relation for Fermat numbers, we can write

$$a^{2^n}+1 = (a-1)(a^{2^0}+1)(a^{2^1}+1)\cdots(a^{2^{n-1}}+1)+2,$$

hence $\gcd(a^{2^n}+1, a^{2^k}+1) \mid 2$ for all $0 \leq k < n$. Since $a^{2^n}+1$ is a power of 2 at best when $n=0$, each term $a^{2^n}+1$ introduces some new prime factor.

Alternatively, assume there are finitely many such primes. Let $M$ be their product. Then $\gcd(M, a^{\varphi(M)}+1) \mid 2$, since $\gcd(a^{\varphi(M)}-1, a^{\varphi(M)}+1) \mid 2$ and $M \mid a^{\varphi(M)}-1$. So $a^{\varphi(M)}+1$ has odd prime factors not among those of $M$.

Yet other alternative solution(s) could work by invoking Euler's criterion, the quadratic reciprocity law, or else the theorems of Zsigmondy or Kobayashi.

ii) If $p$ is a prime such that $-1$ and $-a$ are non-quadratic residues modulo $p$, then $p$ divides no term $a^n+1$, for any $n \geq 1$. This is since, for $n = 2m$, $p \mid a^{2m}+1$ is equivalent to $(a^m)^2 \equiv -1 \pmod{p}$, forbidden, while for $n = 2m-1$, $p \mid a^{2m-1}+1$ is equivalent to $(a^m)^2 \equiv -a \pmod{p}$, also not allowed. Assume finitely many such primes $q_1, q_2, \ldots, q_k$ do exist, and consider the number $N = 4a(q_1q_2\cdots q_k)^2 - 1$.

Clearly any prime $q \mid N$ is distinct from all of them. Since $N \equiv -1 \pmod{4}$, at least one of its prime factors is $q \equiv -1 \pmod{4}$, thus with $-1$ a non-quadratic residue modulo $q$. On the other hand, then $(2aq_1q_2\cdots q_k)^2 \equiv a \pmod{q}$, thus $a$ is a quadratic residue modulo $q$, and so $-a$ will be a non-quadratic residue. This contradicts the fact that all such primes were assumed to be contained in $\{q_1, q_2, \ldots, q_k\}$.

Other, alternative solution(s), could work by invoking the quadratic reciprocity law. The very same conclusion on the infinity of such primes could be obtained by some direct application of Dirichlet's Theorem. ■

**Remarks.** A more palatable version, for $a=3$, $n \mapsto 2^n$, asked in the Juniors' paper, allows for quite more specific solutions, due to the precision gained on the value of $a$.

**Problem 4.** Given a (fixed) positive integer $N$, solve the functional equation

$$f\colon \mathbb{Z} \to \mathbb{R},\ f(2k) = 2f(k) \text{ and } f(N-k) = f(k), \text{ for all } k \in \mathbb{Z}.$$

DAN SCHWARZ

**Solution.** The claim is the only such function is the zero function, $\boxed{f(k)=0}$ for all $k \in \mathbb{Z}$, which trivially fulfills the requirements.

Tackle first the case of odd $N$. Consider the following *directed* graph $G$. Its vertices are the integers $v \in \mathbb{Z}$. Its edges are made by red arrows $2k \to k$ and by blue arrows $2k+1 \to N-(2k+1)$, for all $k \in \mathbb{Z}$. Then for each vertex, its out-degree is exactly 1 (from an even vertex does leave a red arrow, while from an odd vertex leaves a blue arrow). Also define $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ so that $\varphi(v)$ is the end of the arrow starting at $v$. Start now with any vertex $v$, and do build the unique directed path $P_v = v \to \varphi(v) \to \varphi(\varphi(v)) \to \cdots \to \varphi^n(v) \to \cdots$, given by the iterates of $\varphi$ computed at $v$.

1. It is readily seen that for $1 \leq v \leq N-1$ we also have $1 \leq \varphi(v) \leq N-1$, so if we start with $1 \leq v \leq N-1$, the path $P_v$ is contained in the induced subgraph $G' = G \cap \{1, 2, \ldots, N-1\}$. Since this graph is finite, the path will eventually reach some vertex $w$ already passed through, thereby creating a cycle. Now, that means $f(w) = 2^r f(w)$, where $r \geq 1$ represents the number of red arrows in the cycle, therefore $f(w) = 0$. But we also have $f(v) = 2^\rho f(w)$, where $\rho \geq 0$ is the number of red arrows in the path from vertex $v$ to vertex $w$, therefore $f(v) = 0$. Since clearly $f(0) = 2f(0)$, so $f(N) = f(0) = 0$, it follows $f(v) = 0$ for all $0 \leq v \leq N$.

2. If $v > N$ or $v < 0$, then $\left|\varphi(v) - \frac{1}{2}N\right| < \left|v - \frac{1}{2}N\right|$ when $v$ is even. When $v$ is odd, then $\varphi(v) = N - v$, hence $\left|\varphi(v) - \frac{1}{2}N\right| = \left|(N-v) - \frac{1}{2}N\right| = \left|v - \frac{1}{2}N\right|$. Since the path $P_v$ cannot contain two consecutive odd vertices, it follows that $\left|\varphi(\varphi(v)) - \frac{1}{2}N\right| < \left|v - \frac{1}{2}N\right|$. Therefore, via the principle of infinite descent, there will be a moment when a vertex $w$ on the path will have $\left|w - \frac{1}{2}N\right| \leq \frac{1}{2}N$, hence $0 \leq w \leq N$, and so $f(w) = 0$. Now, that means $f(v) = 2^\rho f(w)$, where $\rho > 0$ is the number of red arrows in the path from vertex $v$ to vertex $w$, therefore $f(v) = 0$.[1]

For even $N = 2^m M$, with $m \geq 1$ and odd $M$, focus on the numbers $2^m v$. Let $g\colon \mathbb{Z} \to \mathbb{R}$ defined by $g(x) = f(2^m x)$. Then $g(2k) = f(2^{m+1}k) = 2f(2^m k) = 2g(k)$, and $g(k) = f(2^m k) = f(N - 2^m k) = f(2^m(M-k)) = g(M-k)$. Therefore we are back to the problem case, for $M$ instead of $N$ and $g$ instead of $f$. Since $M$ is odd, according with the above it follows $g$ is the null function, thus $f$ is zero on the numbers $2^m v$, and since for all $v$ we have $f(v) = f(2^m v)/2^m$, it follows $f$ is also zero on all numbers.[2] ■

**Remarks.** We present a few cases for the main cycles of the graph. Call *dendrite* a path ending in a vertex of some cycle. Then we can analyze the following cases

$\boxed{N=7}$ 1→6→3→4→2→1 (with dendrite 5→2); therefore connected;

$\boxed{N=17}$ 1→16→8→4→2→1 (with dendrites 9→8, 13→4, 15→2), and 3→14→7→10→5→12→6→3 (with last dendrite 11→6); therefore disconnected.

*Lemma.* Given an odd $N$, the dendrites are just those odd numbers $v$ with $\frac{1}{2}N < v < N$, i.e. those with $2v > N$, whose in-degrees are equal to 0.[3]

*Proof.* Just remove these vertices from $G'$; the subjacent undirected graph is 2-regular (with one red and one blue edge each incident with each vertex), therefore a union of disjoint cycles. The other vertices are connected each to one different even vertex in some cycle, by a blue edge. □

★ ★ ★ **END**

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[1] Notice the graph $G'$ can be disconnected (always like this for composite $N$'s, and also for some particular prime $N$'s, like $17, 31, \ldots, 73, \ldots, 127, \ldots$). Can we characterize precisely when? I guess not! as it derives from the binary structure of $N$.

[2] The issue now of the graph $G'$ being connected or not boils down to the connectedness of the graph for $M$, all the other vertices being connected to its cycle(s) by *dendrites* (i.e. paths ending in a vertex of some cycle).

[3] The structure for $N = 2^m M$ is derived from that of the odd $M$ factor. For example, for $M = 2013 = 3\cdot 11\cdot 61$, the structure of the cycles is given by the factors $3, 11, 61$, which are all Hamiltonian (so all in all we will have $2^3 - 1 = 7$ disjoint cycles – this is based on a **conjecture** that the cycle structure for any square-free odd number $M$ has $2^{\omega(M)} - 1 = \tau(M) - 1$ cycles, where $\omega(M)$ is here the number of prime factors of $M$ and $\tau(M)$ is the number of positive divisors of $M$, provided each such prime is proved to be Hamiltonian; a related **conjecture** is that for such a prime $p$ then $p^m$ has $m$ cycles).