Clasa a X-a - Etapa 3

Problema 1. *Arătaţi că:*

$$|z+w| \geq \frac{1}{2}\left(\,|z|+|w|\,\right)\left|\frac{z}{|z|}+\frac{w}{|w|}\right|,\ \forall\, z,\, w \in \mathbb{C}^*.$$

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Soluţie. Fie $z,\, w \in \mathbb{C}^*$. Avem:

$$|z+w|^2 - \frac{1}{4}\left(\,|z|+|w|\,\right)^2\left|\frac{z}{|z|}+\frac{w}{|w|}\right|^2$$

$$= |z|^2+|w|^2+2\mathrm{Re}(\bar{z}w) - \frac{1}{4}\left(|z|^2+|w|^2+2|z|\cdot|w|\right)\left(2+2\frac{\mathrm{Re}(\bar{z}w)}{|z|\cdot|w|}\right)$$

$$= |z|^2+|w|^2+2\,\mathrm{Re}(\bar{z}w) - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 - |z|\cdot|w| - \frac{1}{2}\left(|z|^2+|w|^2+2|z|\cdot|w|\right)\frac{\mathrm{Re}(\bar{z}w)}{|z|\cdot|w|}$$

$$= \frac{1}{2}\left(|z|^2+|w|^2-2|z|\cdot|w|\right)+2\frac{\mathrm{Re}(\bar{z}w)}{|z|\cdot|w|}|z||w| - \frac{1}{2}\left(|z|^2+|w|^2+2|z|\cdot|w|\right)\frac{\mathrm{Re}(\bar{z}w)}{|z|\cdot|w|}$$

$$= \frac{1}{2}\left(\,|z|-|w|\,\right)^2 - \frac{1}{2}\left(|z|^2+|w|^2-2|z|\cdot|w|\right)\frac{\mathrm{Re}\,(\bar{\mathrm{z}}\mathrm{w})}{|z|\cdot|w|}$$

$$= \frac{1}{2}\left(\,|z|-|w|\,\right)^2\left(1-\frac{\mathrm{Re}(\bar{z}w)}{|z|\cdot|w|}\right) \geq 0,$$

deoarece $\mathrm{Re}(\bar{z}w) \leq |\bar{z}w| = |\bar{z}|\cdot|w| = |z|\cdot|w|$.

Egalitatea are loc pentru $|z|=|w|$ sau $\mathrm{Re}(\bar{z}w) = |z|\cdot|w|$.