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Problem 1. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+6 \geq 2\sqrt{2}\left(\sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}}\right).$$

When does equality hold?

Solution. We have $\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}=\sum \frac{b+c}{a}=\sum \frac{1-a}{a}$, so

$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+6-2\sqrt{2}\left(\sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}}\right)=$$

$$\sum\left(\frac{1-a}{a}-2\sqrt{2}\sqrt{\frac{1-a}{a}}+2\right)=\sum\left(\sqrt{\frac{1-a}{a}}-\sqrt{2}\right)^2 \geq 0.$$

Equality occurs when $\sqrt{\frac{1-a}{a}}=\sqrt{\frac{1-b}{b}}=\sqrt{\frac{1-c}{c}}=\sqrt{2}$, therefore when (the expected) $a=b=c=\frac{1}{3}$. ■

Problem 2. Circles k_1 and k_2 meet at distinct points A and B. Line t is tangent to k_1 and k_2 at points M, respectively N. When $t \perp AM$ and $MN = 2AM$, compute the measure of angle $\angle NMB$.

Solution. Let P be the midpoint of the segment MN. The powers of the point P with respect to the circles k_1 and k_2 are equal, therefore P belongs to the radical axis of the circles k_1 and k_2, which is precisely the line AB, so A, B, P are collinear. Now $\triangle AMP$ is isosceles right-angled, therefore $\angle NMB = \angle MAB = 45°$. Moreover, the point B is the midpoint of the segment AP, and $MB \perp AP$. ■

Problem 3. A number of $n > 1$ nails are pairwise connected with monochromatic ropes coloured using n distinct colours. It is given that, for any three distinct colours, there exist three nails with ropes connecting them coloured by precisely those three colours. Is it possible that

(a) $n = 6$?
(b) $n = 7$?

Solution. Each rope participates in precisely $n-2$ triangles. There are $\binom{n-1}{2} = \dfrac{(n-1)(n-2)}{2}$ triplets of distinct colours containing a fixed colour c, so one needs at least $\left\lceil \dfrac{(n-1)(n-2)/2}{n-2} \right\rceil = \left\lceil \dfrac{n-1}{2} \right\rceil$ ropes of each colour c of the n colours used. Finally, there are $\binom{n}{2} = \dfrac{n(n-1)}{2}$ ropes in all, so it is needed that $n\left\lceil \dfrac{n-1}{2} \right\rceil \leq \dfrac{n(n-1)}{2}$, henceforth n must be odd.

(a) For $n=6$ we thus proved no such colouring is possible.

(b) For $n=7$, in order to have such a colouring, one thus needs 3 ropes of each of the 7 colours used, accounting for precisely all of the 21 ropes, and each of the 5 triangles made with a rope must be tricolour.

A model is obtained as follows. Label both the nails and the colours by the elements of $\mathbb{F}_7 = \{0,1,2,3,4,5,6\}$. Assign to rope ij the colour $i+j \pmod 7$. It is trivial that each triangle ijk (of the 35 possible) is tricolour (of the 35 possible combinations of 3 colours out of 7). ■

Remark. Notice the above holds in general; there is no solution for n even, and a model for n odd is given by assigning to rope ij the colour $i+j \pmod n$. As pointed out on AoPS, this precise general question was Problem 2, Day 2, China National Olympiad 2009.

Problem 4. Determine all positive integers x, y, z, t for which

$$2^x \cdot 3^y + 5^z = 7^t.$$

Solution. Modulo 3 we must have $(-1)^z \equiv 5^z \equiv 7^t \equiv 1 \pmod 3$, so we must have $\boxed{z \text{ even}}$. Modulo 7 we must have $2^x \cdot (-4)^y \equiv -(-2)^z \pmod 7$, thus $(-1)^y 2^{x+2y} \equiv -2^z \pmod 7$, that is $2^{x+2y-z} \equiv -(-1)^y \pmod 7$, which requires $\boxed{y \text{ odd}}$, since $2^m \equiv -1 \pmod 7$ has no solution.

Let us first decide on $x=1$. Modulo 4 we must have $2 \cdot 3^y \equiv -1 + (-1)^t \pmod 4$, therefore we must have t odd. Then modulo 5 we must have $2 \cdot (-2)^y \equiv 2^t \pmod 5$, thus $2^{1+y-t} \equiv -1 \pmod 5$, impossible since $1-y+t$ is odd, while $2^m \equiv -1 \pmod 5$ requires m even.

So we continue with $x \geq 2$. Modulo 4 we must now have $1 \equiv (-1)^t \pmod 4$, therefore we must have $\boxed{t = 2t' \text{ even}}$. Remember $z = 2z'$ is also even, so $2^x \cdot 3^y = (7^{t'} - 5^{z'})(7^{t'} + 5^{z'})$. But $\gcd(7^{t'} - 5^{z'}, 7^{t'} + 5^{z'}) = 2$. Now, if $3 \mid 7^{t'} - 5^{z'}$, then we must have $2 \cdot 3^y + 5^{z'} = 7^{t'}$ (since $7^{t'} + 5^{z'} > 2$), which is the impossible case of the above.[1] Therefore we need consider $7^{t'} - 5^{z'} = 2^{x'}$ (with, in parallel, $7^{t'} + 5^{z'} = 2^{x-x'} \cdot 3^y$), where $x' = 1$ or $x' = x-1$.

[1] The consideration of the case $x=1$ now acts in some way as a step in an *infinite descent* method of argumentation, leading to focusing on the only other possibility, as that in the sequel.

After this effort, we enter a new phase. Modulo 3 we now need have $7^{t'} - 5^{z'} \not\equiv 0 \pmod 3$, whence z' odd. But modulo 16 we do have $2^x \cdot 3^y = 49^{t'} - 25^{z'} \equiv 8 \pmod{16}$, and this forces $\boxed{x = 3}$. One obvious possibility is $t' = z' = x' = 1$, with the solution $\boxed{2^3 \cdot 3^1 + 5^2 = 7^2}$.

The case $7^{t'} - 5^{z'} = 2^2$ cannot occur, since modulo 3 we would have $2 \equiv 7^{t'} - 5^{z'} = 2^2 \equiv 1 \pmod 3$, absurd; so the only case left is $7^{t'} - 5^{z'} = 2$ (with $7^{t'} + 5^{z'} = 2^2 \cdot 3^y$).[2] Write $7^{t'} - 5^{z'} = 2 = 7 - 5$, or $7(7^{t'-1} - 1) = 5(5^{z'-1} - 1)$. Since the order of 7 modulo 5 is 4, it means that $4 \mid t' - 1$, and then $2^5 \cdot 3 \cdot 5^2 = 7^4 - 1 \mid 7^{t'-1} - 1$, so $5^2 \mid 5(5^{z'-1} - 1)$, only possible when $z' = 1$ and $t' = 1$, which is the boxed solution above. ■

Remark. I do not particularly relish this type of problem; with sterile manipulations galore modulo carefully chosen numbers, in the chase for some contradiction. As a matter of strategy, the only reasonable way of attack is to somehow factorize $7^t - 5^z$ (and not some other combination, since the term $2^x \cdot 3^y$ contains two unknowns); thus one is led to reaching the necessary condition that both z and t must be even, and then using the difference of squares, as in the solution above.

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[2] We can kill it with Lifting The Exponent lemma. We have $7^{t'} - 1 = 2 \cdot 3^y$. Since $v_3(7-1) = 1$, and by LTE $v_3(7^{t'} - 1) = 1 + v_3(t') = y$, we will need $3^{y-1} \mid t'$, but then, for $y > 1$, we would have $7^{t'} - 1 > 2 \cdot 3^y$. Therefore $y = 1$, and the only solution is the one boxed above.