Clasa a X-a – Etapa 3 - Problema 4

Enunţ: *Fie* $x, y, z \in \mathbb{C}$ *cu* $|x| = |y| = |z| = 1$. *Demonstraţi că*

$$|x-y|+|y-z|+|z-x| \geq |x-y| \cdot |y-z| \cdot |z-x|.$$

Soluţie. Fie $\alpha, \beta, \gamma \in \mathbb{R}$ şi $x, y, z, u \in \mathbb{C}$. Atunci are loc egalitatea

$$\left|(\alpha+\beta+\gamma)u-(\alpha a+\beta b+\gamma c)\right|^2 = (\alpha+\beta+\gamma)\left(\alpha|u-x|^2+\beta|u-x|^2+\gamma|u-z|^2\right) - \alpha\beta|x-y|^2-\alpha\gamma|x-z|^2-\beta\gamma|y-z|^2,$$

vezi articolul: **N. Bourbăcuţ, O identitate cu numere complexe şi consecinţele sale geometrice, Recreaţii Matematice, 2011.**

Atunci obţinem

$$(\alpha+\beta+\gamma)\left(\alpha|u-x|^2+\beta|u-y|^2+\gamma|u-z|^2\right) \geq \alpha\beta|x-y|^2+\alpha\gamma|x-z|^2+\beta\gamma|y-z|^2.$$

Alegem $u=0$ şi avem

$$(\alpha+\beta+\gamma)^2 \geq \alpha\beta|x-y|^2+\alpha\gamma|x-z|^2+\beta\gamma|y-z|^2.$$

Fie $\alpha=|y-z|$, $\beta=|x-z|$ şi $\gamma=|x-y|$ şi obţinem concluzia.