Etapa 7, Problema 2

Demonstrați că $\frac{2}{n!(n+2)!} < \prod_{k=1}^{n}\left(\sqrt[k+1]{\frac{k+1}{k}}-1\right) < \frac{1}{(n+1)(n!)^2}$, oricare ar fi numărul natural nenul *n*.

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Soluție.

Folosind inegalitatea mediilor, deducem că

$$\sqrt[k+1]{\frac{k+1}{k}}-1=\sqrt[k+1]{\left(1+\frac{1}{k}\right)\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{k}}-1<\frac{1+\frac{1}{k}+k}{k+1}-1=\frac{1}{k(k+1)}$$

$$\Rightarrow \prod_{k=1}^{n}\left(\sqrt[k+1]{\frac{k+1}{k}}-1\right)<\prod_{k=1}^{n}\frac{1}{k(k+1)}=\frac{1}{n!(n+1)!}=\frac{1}{(n+1)(n!)^2}.$$

Pentru a doua inegalitate, avem:

$$\sqrt[k+1]{\frac{k+1}{k}}-1=\sqrt[k+1]{\left(\frac{k+1}{k}\right)\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{k}}-1>\frac{k+1}{\frac{k}{k+1}+k}-1=\frac{(k+1)^2}{k^2+2k}-1=\frac{1}{k(k+2)}.$$

Atunci $\prod_{k=1}^{n}\left(\sqrt[k+1]{\frac{k+1}{k}}-1\right)>\prod_{k=1}^{n}\frac{1}{k(k+2)}=\frac{2}{n!(n+2)!}$ și, de aici, concluzia problemei.