Etapa 2, Problema 1

Pentru $n \in \mathbb{N}$, $n \geq 2$, considerăm numărul complex $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Demonstraţi că, oricare ar fi $z \in \mathbb{C}$ cu $|z| = 1$, are loc egalitatea

$$|z-1|^2 + |z-\omega|^2 + \left|z-\omega^2\right|^2 + \ldots + \left|z-\omega^{n-1}\right|^2 = 2n.$$

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Soluţie.

Folosind relaţiile $\omega\overline{\omega} = 1$, $\omega^n = 1$ şi $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$, obţinem:

$$\begin{aligned}
\sum_{k=1}^{n} \left|z-\omega^k\right|^2 &= \sum_{k=1}^{n} \left(z-\omega^k\right)\left(\overline{z}-\overline{\omega}^k\right) \\
&= \sum_{k=1}^{n} z\overline{z} - z\sum_{k=1}^{n} \overline{\omega}^k - \overline{z}\sum_{k=1}^{n} \omega^k + \sum_{k=1}^{n} (\omega\overline{\omega})^k \\
&= n - 0 - 0 + n = 2n.
\end{aligned}$$