The problems (with solutions) of the second EGMO, April 8-14, 2013, Luxembourg

This material mostly presents original solutions (not the original official solutions – with a couple of notable exceptions). Some comments will shed even more light on the hidden sides of the problems.

Problem 1. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$.

Prove that, if $AD = BE$, then the triangle ABC is right-angled.

proposed by UK - David Monk

Solution (synthetic). (Emil Kolev) Let F be the midpoint of segment AE and M be the midpoint of segment AB. Then $MC = \frac{1}{2}AD = \frac{1}{2}BE = MF$, since they are mid-lines in the triangles ABD, respectively ABE. Thus triangle CMF is isosceles at M, with MA as a median, therefore also an altitude, whence $\angle BAC = \frac{\pi}{2}$. ■

Solution (metric). (Dan Schwarz) We have

$$BE^2 = a^2 + (3b)^2 - 2a(3b)\cos C = a^2 + 9b^2 - 6ab\cos C$$

$$AD^2 = a^2 + b^2 - 2ab\cos(\pi - C) = a^2 + b^2 + 2ab\cos C$$

by the cosine theorem in triangles BCE and ACD. Subtracting the two yields $8b^2 = 8ab\cos C$, thus $\cos C = \frac{b}{a}$, forcing $\angle BAC = \frac{\pi}{2}$.

One of the official solutions starts like this, but then unnecessarily complicates itself; it seems that this was a common ailing on this problem, since the Romanian competitors also have chosen more complicated approaches, mostly based on the formula for the median. ■

Problem 2. Determine all integers m for which the $m\times m$ square can be dissected into five rectangles, the side lengths of which are the integers $1, 2, 3, \ldots, 10$ in some order.

proposed by Finland

Solution. (Dan Schwarz) We first do a sketchy analysis of how the five rectangles may be positioned. Trying to dissect any rectangle into rectangles with no sides equal, it is easily seen this cannot be done with two such rectangles; nor with three, since one must then cover two corners of the large rectangle; nor with four. Finally, the only way with five is to have one interior, of dimensions $x \times y$ parallel to those of the large rectangle, surrounded by four others, of dimensions $(x + v) \times z$, $u \times (y + z)$, $(x + u) \times t$ and $v \times (y + t)$. Because of the symmetry of notations, we may suppose $x < y$, $z < t$ and $u < v$. Indeed the models that will be produced are unique up to permuting (u, v) and/or (z, t), or altogether the order of the dimensions for all rectangles.

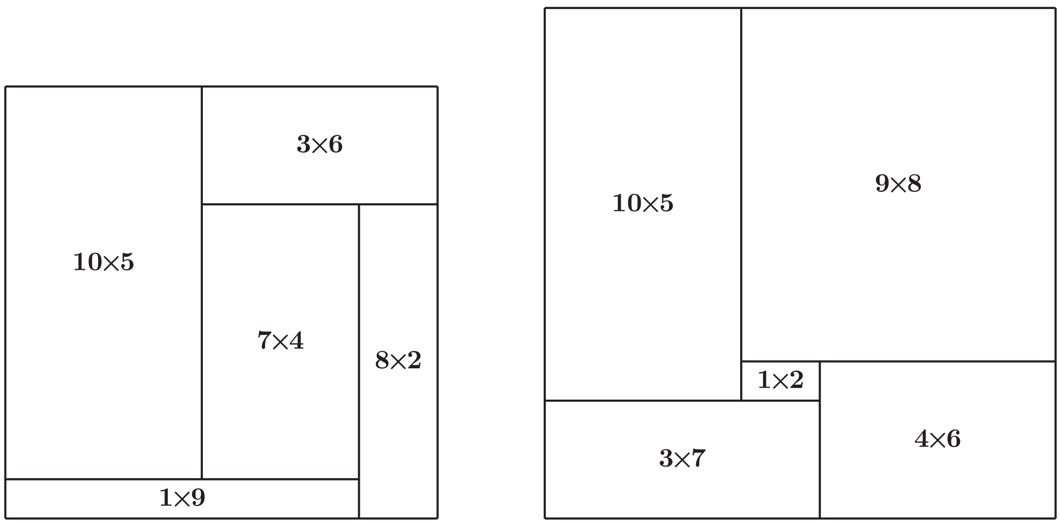
Thus we should have $3(x+y)+2(z+t+u+v)=55$, whence $A=x+y\geq 3$ must be odd, and $B=z+t+u+v\geq 10$. Also, since $m=x+u+v=y+z+t=(A+B)/2$, we must also have B odd. Under these conditions, the Diophantine equation $3A+2B=55$ has as only solutions $(A,B)\in\{(3,23),(7,17),(11,11)\}$, corresponding to $m\in\{13,12,11\}$.

But for $m=11$ we must have $B=11$, and this is only possible for $\{z,t,u,v\}=\{1,2,3,5\}$, thus $x=4=z+t$ (the least unused value), whence $z=1$ and $t=3$, and then $y=7=u+v$, with $u=2$ and $v=5$.

For $m=12$ we must clearly have $z,t,u,v\geq 2$, so $x=1$ (the least unused value), thus $y=6$ (from $x+y=A=7$). Therefore $z+t=6$, so $z=2$ and $t=4$; on the other hand then $u+v=11$, so $u=3$ and $v=8$, but $y+z=8$, repeating the value 8. No solution here.

Finally, for $m=13$ we must have $A=x+y=3$, so $x=1$ and $y=2$. Now, if $z=3$, this leads to $t=8$, but also $u=4$ and $v=8$, repeating the value 8, so no solution. However, if $u=3$, this leads to $v=9$, then $z=5$ and $t=6$.

The official solution starts by establishing bounds on m in different possible ways, then exhibiting models for $m=11$ and $m=13$ by "trial and error", finally ruling out the case $m=12$. ■



Problem 3. Let n be a positive integer.

(a) Prove that there exists a set S of $6n$ pairwise different positive integers, such that the least common multiple of any two elements of S is no larger than $32n^2$.

(b) Prove that every set T of $6n$ pairwise different positive integers contains two elements the least common multiple of which is larger than $9n^2$.

proposed by Romania - Dan Schwarz, after an idea by Pál Erdös

Solution.

(a) Consider the set $S = \{1, 2, \ldots, 4n\} \cup \{4n+2k \mid 1 \leq k \leq 2n\}$, made of $6n$ positive integers. The least common multiple of any two of its elements is then at most $8n(4n-1) = 32n^2 - 8n < 32n^2$.

(b) Let $1 \leq a_1 < a_2 < \cdots < a_{6n}$ be those integers in the set T. Then clearly $a_k \geq k$ for all $1 \leq k \leq 6n$. So we have $a_k \geq m$ for all $m \leq k \leq 6n$ and some $1 \leq m < 6n$, thus $\dfrac{1}{a_{6n}} \leq \dfrac{1}{a_k} \leq \dfrac{1}{m}$. Partition the interval $\left[\dfrac{1}{a_{6n}}, \dfrac{1}{m}\right]$ into $6n - m$ equal-length subintervals. By the pigeonhole principle, there do exist indices $m \leq i < j \leq 6n$ such that $\dfrac{1}{a_i}$ and $\dfrac{1}{a_j}$ both belong to a same subinterval (since there are $6n - m + 1$ such fractions, but only $6n - m$ subintervals), hence

$$0 < \frac{1}{a_i} - \frac{1}{a_j} \leq \frac{1}{6n-m}\left(\frac{1}{m} - \frac{1}{a_{6n}}\right) < \frac{1}{m(6n-m)}.$$

But $\dfrac{1}{a_i} - \dfrac{1}{a_j}$ is a positive fraction of denominator $\operatorname{lcm}[a_i, a_j]$, whence $\operatorname{lcm}[a_i, a_j] > m(6n-m)$. The largest value in the right-hand side is obtained for $m = 3n$, when we get $\operatorname{lcm}[a_i, a_j] > 9n^2$. ■

Problem 4. Find all positive integers a and b for which there are three consecutive integers at which the polynomial

$$P(n) = \frac{n^5 + a}{b}$$

takes integer values.

proposed by Slovenia

Solution. (Dan Schwarz) Let us then have b divide $(n-1)^5 + a$, $n^5 + a$ and $(n+1)^5 + a$ for some integer n. It follows

$$b \mid (n+1)^5 - n^5 = 5n^4 + 10n^3 + 10n^2 + 5n + 1,$$
$$b \mid n^5 - (n-1)^5 = 5n^4 - 10n^3 + 10n^2 - 5n + 1.$$

By adding and subtracting the relations above we get $b \mid 10n(2n^2+1)$ and $b \mid 10n^4 + 20n^2 + 2$. But $\gcd(b, 2) = \gcd(b, n) = 1$, so $b \mid 5(2n^2 + 1)$.

On the other hand $b \mid 2(10n^4 + 20n^2 + 2) = 10n^2(2n^2+1) + 15(2n^2+1) - 11$, hence $b \mid 11$. This immediately offers the solutions $\boxed{(a, b) = (m, 1), \text{ for all } m \in \mathbb{N}^*}$.

For $b = 11$ we need have $a \equiv -(n-1)^5 \pmod{11}$, $a \equiv -n^5 \pmod{11}$ and $a \equiv -(n+1)^5 \pmod{11}$. But the values of $k^5 \pmod{11}$ for $k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are $0, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1$, so we need $a \equiv \pm 1 \pmod{11}$, and so the extra solutions $\boxed{(a, b) = (11m \pm 1, 11), \text{ for all } m \in \mathbb{N}^*}$, together with $\boxed{(a, b) = (1, 11)}$.

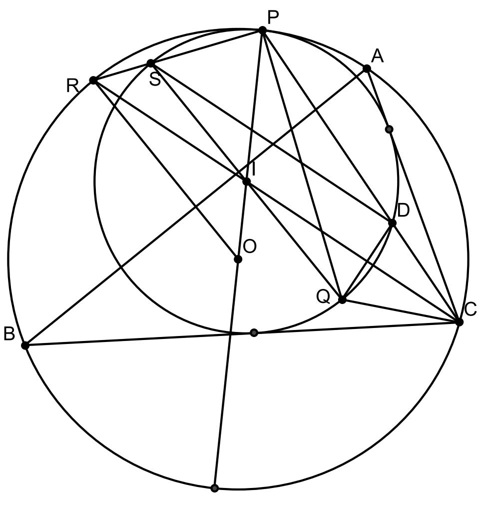
In fact, by the Euclidean algorithm, the polynomial greatest divisor of $5x^4 + 10x^3 + 10x^2 + 5x + 1$ and $5x^4 - 10x^3 + 10x^2 - 5x + 1$ is 22, and we have $\gcd(b, 2) = 1$. ■

Problem 5. Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB and intersecting the interior of triangle ABC is tangent to ω at Q.

Prove that $\angle ACP = \angle QCB$.

proposed by Poland - Waldemar Pompe

Solution. (Csapó Hajnalka) Denote by O the centre of Ω and by I the centre of ω. From the givens it follows points P, O and I are collinear, $QI \perp AB$ and CI is the angle bisector of $\angle ACB$. Taking $\{D\} = CP \cap \omega$, $\{S\} = QI \cap \omega$ and $\{R\} = CI \cap \Omega$, it follows R is the midpoint of the arc AB, hence $OR \perp AB$. Also $IS \perp AB$. Thus $\angle PIS \equiv \angle POR$; but $\angle PIS = 2\angle PDS$, while $\angle POR = 2\angle PCR$, whence $\angle PDS \equiv \angle PCR$, so $DS \parallel CR$. On the other hand, $DS \perp DQ$ (SQ is diameter in circle ω), hence $CR \perp DQ$, therefore CI is the perpendicular bisector of the segment DQ, and so CI is the angle bisector of $\angle DCQ$. Since CI is also the angle bisector of $\angle ACB$, it follows $\angle ACP \equiv \angle QCB$ (in other words, the lines CP and CQ are isogonal). ■



Remarks. By Archimedes' Lemma it follows points P, R and S are collinear (triangles PIS and POR are similar), thus by the homothety of centre P which sends ω onto Ω, point S is sent onto R and point D is sent onto C, therefore $DS \parallel CR$, another way to reach this seminal result.

Problem 6. Snow White and the Seven Dwarves are living in their house in the forest. On each of 16 consecutive days, some of the dwarves worked in the diamond mine while the remaining dwarves collected berries in the forest. No dwarf performed both types of work on the same day. On any two different (not necessarily consecutive) days, at least three dwarves each performed both types of work. Further, on the first day, all seven dwarves worked in the diamond mine.

Prove that, on one of these 16 days, all seven dwarves were collecting berries.

proposed by Bulgaria - Emil Kolev

Solution (official). We define V as the set of all $2^7 = 128$ vectors of length 7 with entries in $\{0, 1\}$. Every such vector encodes the work schedule of a single day – if the i-th entry is 0 then the i-th dwarf works in the mine, and if this entry is 1 then the i-th dwarf collects berries. The 16 working days correspond to 16 vectors $d_1, \ldots, d_{16}$ in V, which we will call *day-vectors*. The condition on any pair of distinct days means that any two distinct day-vectors d_i and d_j differ in at least three positions.

We say that a vector $x \in V$ *covers* some vector $y \in V$ if x and y differ in at most one position; note that every vector in V covers exactly eight vectors. For each of the 16 day-vectors d_i we define $B_i \subset V$ as the set of the eight vectors that are covered by d_i. Since for $i \neq j$ the day-vectors d_i and d_j differ in at least three positions, their corresponding sets B_i and B_j are disjoint. As the sets $B_1, \ldots, B_{16}$ together contain $16 \cdot 8 = 128 = |V|$ distinct elements, they form a partition of V; in other words, every vector in V is covered by precisely one day-vector.

The *weight* of a vector $v \in V$ is now defined as the number of entries equal to 1 in v. For $k = 0, 1, \ldots, 7$, the set V contains $\binom{7}{k}$ vectors of weight k. Let us analyze the 16 day-vectors $d_1, \ldots, d_{16}$ by their weights, and let us discuss how the vectors in V are covered by them.

1. As all seven dwarves work in the diamond mine on the first day, the first day-vector is given by $d_1 = (0, 0, 0, 0, 0, 0, 0)$. This day-vector covers all vectors in V with weight 0 or 1.

2. No day-vector can have weight 2, as otherwise it would differ from d_1 in at most two positions. Hence each of the $\binom{7}{2} = 21$ vectors of weight 2 must be covered by some day-vector of weight 3. As every vector of weight 3 covers three vectors of weight 2, exactly $21/3 = 7$ day-vectors have weight 3.

3. How are the $\binom{7}{3} = 35$ vectors of weight 3 covered by the day-vectors? Seven of them are day-vectors, and the remaining 28 ones must be covered by day-vectors of weight 4. As every vector of weight 4 covers four vectors of weight 3, exactly $28/4 = 7$ day-vectors have weight 4.

To summarize, one day-vector has weight 0, seven have weight 3, and seven have weight 4. None of these 15 day-vectors covers any vector of weight 6 or 7, so that the eight heavy-weight vectors in V must be covered by the only remaining day-vector; and this remaining vector must be $(1, 1, 1, 1, 1, 1, 1)$. On the day corresponding to $(1, 1, 1, 1, 1, 1, 1)$, all seven dwarves are collecting berries, and that is what we wanted to show.

Up to permutations of the dwarves, there exists a unique set of day-vectors that satisfies the conditions of the problem statement

0000000	1110000	1001100	1000011	0101010	0100101	0010110	0011001
1111111	0001111	0110011	0111100	1010101	1011010	1101001	1100110

All this is more or less Code Theory techniques. The Hamming distance between two vectors $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$ is defined as the number $\mathrm{d}(x, y)$ of differences between x_k and y_k over all $1 \leq k \leq N$. Since $\mathrm{d}(x, y) = 0$ if and only if $x = y$, $\mathrm{d}(x, y) = \mathrm{d}(y, x)$, and $\mathrm{d}(x, y) + \mathrm{d}(y, z) \geq \mathrm{d}(x, z)$, this is indeed a *distance*. So $B_i = \{x \mid \mathrm{d}(d_i, x) \leq 1\}$, and $B_i \cap B_j = \emptyset$ for all $i \neq j$, since the Hamming distance between day-vectors is at least 3. The 7 day-vectors of weight 3 form in fact the symmetric BIBD $(7, 3, 1)$, representing the Fano finite projective plane of order 2. ■

Solution. (Dan Schwarz) For $n = 3$ let us look at the 2^{n+1} days as being $(2^n - 1)$-dimensional real vectors, with their k-th coordinate representing the k-th dwarf, and of value $+1$ for working in the mine and -1 for collecting berries, for $1 \leq k \leq 2^n - 1$. The conditions of the problem state that any two vectors disagree in at least $2^{n-1} - 1$ coordinates, and that the first vector is $(1, 1, \ldots, 1)$.

Now, Snow White feels bad for not also be at work, so she decides to go each day to that type of work that an odd number of dwarves are performing that day. This means adding a 2^n-th coordinate (representing Snow White) to the vectors, and it is easy to see that any two of the new vectors disagree in at least 2^{n-1} coordinates, and that the first new vector is again $(1, 1, \ldots, 1)$. The pairwise dot-products of the new vectors are non-positive, so the vectors are at pairwise angles of 90° or more.

In fact, we add to the vector $x = (x_1, x_2, \ldots, x_{2^n-1})$ a 2^n-th coordinate (the parity-check bit) $x_{2^n} = \prod_{k=1}^{2^n-1} x_k$, getting the vector x^+. Now, $\left\langle x^+, y^+ \right\rangle = \langle x, y\rangle + x_{2^n} y_{2^n}$, but $\prod_{k=1}^{2^n} x_k y_k = (x_{2^n} y_{2^n})^2 = 1$, implying $\left\langle x^+, y^+ \right\rangle \leq 0$, since we have $\langle x, y\rangle \leq 1$ from the given conditions.

A folklore result however states the following

> In the Euclidean space $\mathbb{R}^d$, the maximum number of unit vectors at pairwise angles of 90° or more is $2d$; moreover, the only maximal configurations are given by d pairwise orthogonal unit vectors, and their opposites.
>
> (The proof goes by induction on d, the case $d = 1$ being trivial, using Linear Algebra methods; for self-sufficiency we will present a proof in the **Remarks**).

Our new vectors are of equal norm, so that's fine, but they are not just any odd vectors – they are position vectors for vertices of the d-dimensional hypercube $\{-1, +1\}^d$. The largest number of pairwise orthogonal vectors in $\mathbb{R}^d$, made of ± 1 entries, is clearly not larger than the dimension d of the space. The bound can be shown to be tight for d a power of 2, by using Sylvester's construction for Hadamard matrices

$$H_1 = \begin{pmatrix} 1 \end{pmatrix}, \; H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}, \; \ldots, \; H_{2^{n+1}} = \begin{pmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{pmatrix}, \; \ldots$$

Therefore such a set of $2d$ vectors exists whenever a Hadamard matrix H_d does exist, by taking the column vectors of the matrix $(H_d \quad -H_d)$. But H_d exists in the case when d is a power of 2, and indeed in our case $d = 8 = 2^3$.

So the set of the new vectors is a maximal set, thus it contains, together with any vector, its opposite. This makes it contain the vector $(-1, -1, \dots, -1)$, and now sending Snow White back to kitchen duties, the result ensues. Moreover, the model is unique, up to a permutation of the dwarves.

Let us notice that we can build a set of $2^{n+1} - 1$ such vectors, containing the vector with all $+1$ entries, but without containing the vector with all -1 entries. It is enough to take the model $(H_{2^n} \quad -H_{2^n})$ of above for 2^{n+1} such vectors in 2^n-th dimension, and remove the vector with all -1 entries, and the 2^n-th coordinate of the remaining $2^{n+1} - 1$ vectors (no resulting vector can be $(-1, -1, \dots, -1)$, since it would have disagreed in at most one position with the removed vector). Therefore a full set of 2^{n+1} days is required for the stated claim to hold; $2^{n+1} - 1$ days are just not enough. ■

Remarks. We need to also mention Hadamard's conjecture that such matrices exist whenever $4 \mid d$ (it is not difficult to show that $4 \mid d$ is a necessary condition, and there are no known counterexamples for it not also being sufficient). For some particular cases, when $d - 1$ is a power of some odd prime $\equiv 3 \pmod 4$, actual models are provided by Paley's construction. The very Hadamard conjecture is in fact attributed to Paley; see `http://en.wikipedia.org/wiki/Hadamard_matrix`.

In order for this solution to be self-sufficient, we will also prove that the largest number of not-null vectors in $\mathbb{R}^d$, of pairwise non-positive dot-products, is $2d$, with full characterization of such maximal sets of vectors.

Let us proceed by simple induction on d, starting with the obvious case $d = 1$. In dimension $d+1$, let u be one of the vectors. Denote by $U = < u >^\perp$ the orthogonal complement of u, of dimension d. Any of the other vectors v can now be uniquely written as $v = v_\perp + v_\dashv$, with $v_\dashv = \dfrac{\langle u, v\rangle}{||u||^2} u$, thus $v_\perp \perp u$, i.e. $v_\perp \in U$. Notice that the coefficients $\dfrac{\langle u, v\rangle}{||u||^2}$ are non-positive under our conditions, and also notice that we can only have $v_\perp = 0$ once, for some $u' = v$ (trivially so, since then $u' = u'_\dashv = \lambda u$ for some negative real λ).

Let us consider next the orthogonal components $v_\perp \in U$, whence $\langle v, w\rangle = \langle v_\perp + v_\dashv, w_\perp + w_\dashv\rangle = \langle v_\perp, w_\perp\rangle + \langle v_\perp, w_\dashv\rangle + \langle v_\dashv, w_\perp\rangle + \langle v_\dashv, w_\dashv\rangle = \langle v_\perp, w_\perp\rangle + \dfrac{\langle u, v\rangle \langle u, w\rangle}{||u||^2}$, and therefore $\langle v_\perp, w_\perp\rangle = \langle v, w\rangle - \dfrac{\langle u, v\rangle \langle u, w\rangle}{||u||^2} \leq 0$, since $\langle v, w\rangle \leq 0$ and $\langle u, v\rangle \langle u, w\rangle \geq 0$. We fall under the induction hypothesis, so the number of the not-null vectors is at most $2d$, therefore (together with u and possibly u') there were at most $2(d+1)$ not-null vectors for dimension $d+1$, and the induction is completed.

In order to build a model, notice further that in order to achieve the bound we must have the pair u, u', with all other vectors orthogonal to them. By inductive reasoning it follows that the only possible model is made by some d pairwise orthogonal not-null vectors, and d other obtained from them by multiplication with arbitrary negative scalars. □

Presentation by Dan Schwarz, April 16, 2013