The 7th "STARS of MATHEMATICS" Competition – Juniors
October 19, 2013 ★★★ ICHB – Bucharest

Solutions

Problem 1. Prove that for any integers a, b, the equation $2abx^4 - a^2x^2 - b^2 - 1 = 0$ has no integer roots.

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Solution. Write the equation as $b^2 - 2ax^4b + a^2x^2 + 1 = 0$. The (reduced) discriminant of this quadratic in b is

$$\Delta = (ax^4)^2 - (a^2x^2 + 1) = a^2x^2(x^6 - 1) - 1 \equiv -1 \pmod{3,4,7},$$

so Δ cannot be a perfect square, since -1 is not a quadratic residue modulo 3 (and neither modulo 4, nor 7).

If we considered the discriminant $\Delta = a^4 + 8ab(b^2 + 1)$ of the initial biquadratic equation (quadratic in x^2) however, this can easily be a perfect square (for example for $a = 4$, $b = 2$), so that may well be a dead-end.

Let us finally notice the same method can be applied, for any integer $n \geq 3$, to the equation $2abx^n - a^2x^2 - b^2 - 1 = 0$, written as $b^2 - 2ax^n b + a^2x^2 + 1 = 0$, as being of discriminant $\Delta = a^2x^2(x^{n-1} - 1)(x^{n-1} + 1) - 1 \equiv -1 \pmod{3,4}$. Alternately, write the equation as $2abx(x^{n-1} + \varepsilon) - \big((ax + \varepsilon b)^2 + 1\big) = 0$ for $\varepsilon \in \{-1, 1\}$ and look modulo 3; for $x \not\equiv 0 \pmod 3$ just take $\varepsilon \equiv -x^{n-1} \pmod 3$. This leads to $(\,\cdot\,)^2 + 1 \equiv 0 \pmod 3$, and so to an impossibility.[1] ■

Problem 2. Three points inside a rectangle determine a triangle. A fourth point is taken inside the triangle. Prove that at least one of the three concave quadrilaterals formed by these four points has perimeter lesser than that of the rectangle.

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Solution. Draw lines, parallel to the sides of the given rectangle, through the vertices of the triangle and consider the least rectangle $ABCD$ thus formed which still contains the triangle. In the general case, one vertex of the triangle is situated at A, while the other two are, say X on the side BC and Y on the side DC (with the possibility that they are even situated at B, C or D). The fourth point Z is inside $\triangle AXY$. Denote by X' the point on side AD such that $XX' \parallel AB$ and by Y' the point on side AB such that $YY' \parallel AD$. Also denote by O the meeting point of XX' and YY'.

The (some of them possibly degenerated) triangles AXX', AYY' and XOY together cover the triangle AXY. Say Z lies in $\triangle XOY$; then its symmetric Z' with respect to the mid-point of XY is contained in the rectangle $ABCD$. But the quadrilateral $AXZ'Y$ is convex, and also contained in the rectangle $ABCD$, thus by a well-known property of plane convex bodies, its perimeter is less than that of $ABCD$, in turn less than that of the original rectangle. Since $ZX = Z'Y$ and $ZY = Z'X$, the perimeter of $AXZY$ is equal to that of $AXZ'Y$, and we are done. The other cases, when Z lies in $\triangle AXX'$ or $\triangle AYY'$ (or both), are treated in a completely similar way. ■

Remarks. An interesting follow-up question has been asked in the Seniors' paper – please check that too.

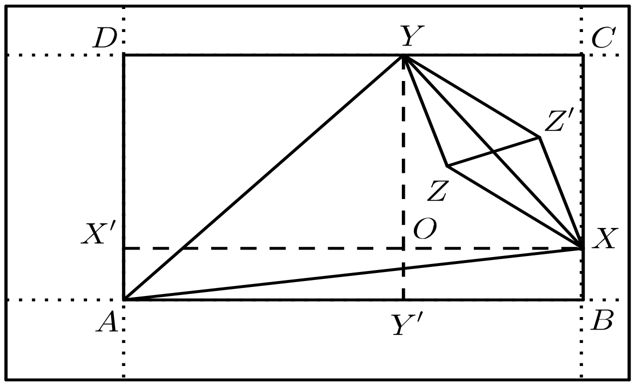

Figure courtesy of ANDREI ECKSTEIN.

Problem 3. Consider the sequence $(3^{2^n} + 1)_{n \geq 1}$.

i) Prove there exist infinitely many primes, none dividing any term of the sequence.

ii) Prove there exist infinitely many primes, each dividing some term of the sequence.

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Solution.
i) By simple inspection for small values of n, it can be seen there appear only prime factors $q \equiv 1 \pmod 4$. This leads to the solution. Consider a prime $p \equiv -1 \pmod 4$. Then $p \mid 3^{2^n} + 1$ is equivalent to $(3^{2^{n-1}})^2 \equiv -1 \pmod p$, impossible, since -1 is a non-quadratic residue modulo p.

But such primes $p \equiv -1 \pmod 4$ are infinitely many. A simple proof follows. Assume finitely many such primes exist, and let M be their product. Consider the number $N = 4M^2 - 1$. Clearly any prime $q \mid N$ is distinct from all of them. On the other hand then $N \equiv -1 \pmod 4$, and so at least one of its factors cannot have residue 1 modulo 4. This contradicts the fact that all of them are contained in M.

The same conclusion on the infinity of such primes could be obtained by a direct application of Dirichlet's theorem.

ii) We have, in a similar vein with the known relation for Fermat numbers,

$$3^{2^n} + 1 = 2(3^{2^0} + 1)(3^{2^1} + 1)\cdots(3^{2^{n-1}} + 1) + 2,$$

hence $\gcd(3^{2^n} + 1, 3^{2^k} + 1) = 2$ for all $0 \leq k < n$. Since $3^{2^n} + 1$ is a power of 2 at best when $n = 0$, each term $3^{2^n} + 1$ introduces some new prime factor.

Equivalently, if $p > 2$ divides $3^{2^n} + 1$ and also $3^{2^k} + 1$ for some $1 \leq k < n$, then $p \mid 3^{2^{k+1}} - 1$, so

$$3^{2^n} + 1 = (3^{2^{k+1}})^{2^{n-k-1}} + 1 \equiv 2 \pmod p,$$

absurd. Therefore $\gcd(3^{2^n} + 1, 3^{2^k} + 1) = 2$, and so, since the halved terms of the sequence are pairwise coprime, there exist infinitely many primes of this second kind.

A quick argument could also go by Zsigmondy's theorem, or by the Kobayashi theorem. ■

Remarks. We would have proved in the above that all primes $p \equiv -1 \pmod 4$ are of the first kind, while some primes $p \equiv 1 \pmod 4$ are of the second kind. What more can be said about the primes $p \equiv 1 \pmod 4$? For some, like 13, we have $13 \mid 3^3 - 1$, thus some are of the first kind. All we can say is that if $p = 2^k q + 1$ is a prime, with q an odd integer and $k \geq 2$, then if p does not divide $3^{2^k} - 1$ it will not divide any term $3^{2^n} + 1$ (so Fermat primes are of the second kind). Is a complete characterisation to be found ? (there is good reason to think – not.)

In a more general setting, and thus with a more general solution, the problem has also been asked in the Seniors' paper – please see that solution too.

Problem 4. A set S of unit cells of an $n \times n$ array, $n \geq 2$, is said *full* if each row and each column of the array contain at least one element of S, but which has this property no more when any of its elements is removed. A full set having maximum cardinality is said *fat*, while a full set of minimum cardinality is said *meagre*.

i) Determine the cardinality $m(n)$ of the meagre sets, describe all meagre sets and give their count.
ii) Determine the cardinality $M(n)$ of the fat sets, describe all fat sets and give their count.

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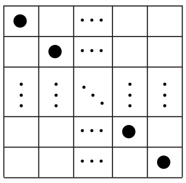
Solution.
i) Clearly, a full set S contains at least n elements; indeed such meagre sets with $|S| = n$ do exist (for good example, the set of cells on the main diagonal of the array; see below). Therefore $\boxed{m(n) = n}$.



Meagre set model.

Any such set has structure $S_\sigma = \{(i, \sigma(i)) \mid 1 \leq i \leq n\}$, for an arbitrary permutation $\sigma \in \mathscr{S}_n$, therefore there exist $\boxed{n! \text{ meagre sets}}$ in all (our example was S_{id}).

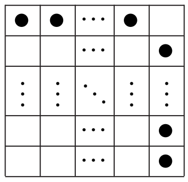
ii) Consider the tripartite graph, having its three classes of vertices being made of the rows $R = \{r_1, r_2, \ldots, r_n\}$, columns $C = \{c_1, c_2, \ldots, c_n\}$, and elements x of S. An edge xr_i exists a if $r_i \cap S = \{x\}$, and an edge xc_j exists if $c_j \cap S = \{x\}$.

The following relations clearly hold

$$1 \leq \deg x \leq 2, \ \deg r_i \in \{0,1\}, \ \deg c_j \in \{0,1\},$$

$$\sum_{i=1}^{n} \deg r_i + \sum_{j=1}^{n} \deg c_j = \sum_{x \in S} \deg x.$$

It follows that $\sum_{x \in S} \deg x \geq \sum_{x \in S} 1 = |S|$. A model S with $2(n-1)$ elements may be obtained by taking the first $n-1$ cells on row r_1 and the last $n-1$ cells on column c_n; see below. We claim $\boxed{M(n) = 2(n-1)}$.



Fat set model.

In order to show no full set S may have more elements, assume $|S| \geq 2n-1$. Since $\sum_{i=1}^{n} \deg r_i + \sum_{j=1}^{n} \deg c_j = \sum_{x \in S} \deg x \geq |S| \geq 2n-1$, it follows that at least one of the two sums in LHS, say the one over the rows, is at least $\lceil (2n-1)/2 \rceil = n$. On the other hand $\sum_{i=1}^{n} \deg r_i \leq \sum_{i=1}^{n} 1 = n$, so $\sum_{i=1}^{n} \deg r_i = n$, with all $\deg r_i = 1$. But this means $2n-1 \leq |S| = n$, absurd for $n \geq 2$. Now, when S has exactly $2(n-1)$ elements, a similar piece of reasoning shows we must have exactly $n-1$ rows and $n-1$ columns of degree 1. For $n \geq 3$ any such set thus has structure

$$S_{i,j} = \{(i,k) \mid 1 \leq k \leq n, k \neq j\} \cup \{(\ell, j) \mid 1 \leq \ell \leq n, \ell \neq i\},$$

for an arbitrary choice $1 \leq i, j \leq n$, therefore there exist $\boxed{n^2 \text{ fat sets when } n \geq 3}$ (our example was $S_{1,n}$). The case $n = 2$ is exceptional, since then $S_{1,1} = S_{2,2}$ and $S_{1,2} = S_{2,1}$, thus there exist only $\boxed{2 \text{ fat sets when } n = 2}$. ■

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[1] I do not know if the equation might ever have rational roots. All I know up to now is that if $x = \dfrac{p}{q}$ (in reduced form) is a root, then we must have $3 \cdot 4 \cdot 7 = 84 \mid q$. An invitation to some bit of research there ...