Tuymaada Mathematical Olympiad 2012

Senior League - Day 1

Problem 1. Tanya and Serezha take turns putting chips in empty squares of a chessboard. Tanya starts with a chip in an arbitrary square. At every next move, Serezha must put a chip in the column where Tanya put her last chip, while Tanya must put a chip in the row where Serezha put his last chip. The player who cannot make a move loses. Whom of the players has a winning strategy?

The fact the setters failed to specify it is an $n \times n$ board, for any $n > 1$, will have some unexpected consequences, as seen below!

(*A. Golovanov*)

Solution. I fear the problem setters blew it big with this one. They forgot to ask for a strategy that works for any $n \times n$ square board! Since the common meaning for *chessboard* is an 8×8 board, let us just notice that Serezha is **forced** to win, whatever he does, on any $n \times n$ board with n even. This is obviously so because, any times he puts a chip, it is on a column where Tanya just put her chip; thus after Serezha plays, each column will still contain an even number of empty squares, so that if Tanya can play, he can also play at his next turn. What a pity ...[1]

Of course, the setters meant it for any $n > 1$. Then, for odd n, Serezha indeed needs a strategy, otherwise he may lose. One strategy (the strategy?) is, after he randomly makes his first move, to always play on the other row that contains chips than the one Tanya just played on. This ensures all chips always occupy only two rows, so whenever Tanya can play, so can Serezha, at his next turn. ■

Alternative Solution. Let us shoot this fly with a big gun. I will make a quite detailed presentation, in order to convince myself it's working ... Let me first start with some trivial observations.

• The game cannot end up in a tie. Trivially so, because eventually the empty squares run out; the game cannot last for more than n^2 moves. A

[1] I know this was the interpretation of at least some competitors, since someone just complained to me about this problem, where "any strategy works, so what the h..k of a strategy is that?".

simple consequence is that either Tanya or Serezha **must** have a winning strategy.[2]

• Say the board is a square $ABCD$ with AB horizontal, such that A is its NW corner and B is its SW corner. Say Tanya sits having side BC in front of her, while Serezha sits having side AB in front of him. Now, from her second turn, Tanya must move on the row Serezha made his last play, while, from his first turn, Serezha must move on the column Tanya made her last play; from his point of view, he moves along a row of the board, as he sees it. Under this interpretation, the game has a definite symmetry in what its rules are concerned.

• The first two moves of Tanya and the first one of Serezha are irrelevant; they create a similar situation on the board. Say then that Tanya starts by putting a chip on the corner A square, denoted by T_1 in the sequel.

Assume now that Tanya has a winning strategy. According to the above, any strategy of hers starts with two random legal moves; we just fixed the first one to be T_1. Now, in the course of her (winning) strategy, the square T_1 is out of play, having been occupied. But then Serezha can *steal* Tanya's strategy.[3] He makes his first move on the corner B square, denoted by S_1 in the sequel, in effect starting with the second move (shown to be arbitrary) of Tanya's (winning) strategy. What Tanya sees now is precisely what Serezha saw when he made his move, only that for her one more other square of the board than S_1 is out of any further play, namely T_1. This means that indeed Serezha can successfully steal Tanya's strategy; he will not be hindered by not being able to make a winning move dictated by this strategy, because that square is out of play, since in fact he plays with one less unavailable square then those Tanya has.

Thus Serezha must win, clearly a contradiction. So Tanya has no winning strategy. Therefore $\boxed{\text{Serezha must have a winning strategy}}$. ■

Remarks. Surely, proving the existence of a non-constructive winning strategy for Serezha is just a theoretical plaything, since exhibiting such a strategy is a quite simple matter, as seen above.

[2] The weak form of Zermelo's theorem, stating that in any finite two-person game of perfect information in which the players move alternatively and in which chance does not affect the decision making process, if the game cannot end in a draw, then one of the two players must have a winning strategy.

[3] John Nash's non-constructive strategy-stealing argument.

Problem 2. Let $P(x)$ be a real quadratic trinomial, such that for all $x \in \mathbb{R}$ the inequality $P(x^3+x) \geq P(x^2+1)$ holds. Find the sum of its roots.

For some reason, the original statement failed to mention $P(x) \in \mathbb{R}[x]$, but mentioned $P(x)$ has two **real** roots; other than suggesting that it has real coefficients, this information is irrelevant towards the solution.

(*A. Golovanov, M. Ivanov, K. Kokhas*)

Solution. Let $P(x) = ax^2+bx+c$. Then $\Delta(x) = P(x^3+x) - P(x^2+1) = P(x(x^2+1)) - P(x^2+1) = (x-1)(x^2+1)(a(x+1)(x^2+1)+b)$. Since we need $\Delta(x) \geq 0$ for all x, we need $x-1 \mid a(x+1)(x^2+1)+b$, i.e. $4a+b=0$. Therefore $b=-4a$, and so by Viète's relation, the sum of the roots of $P(x)$ is $\boxed{x_1+x_2 = -\frac{b}{a} = 4}$. Indeed $\Delta(x) = a(x-1)^2(x^2+1)(x^2+2x+3) \geq 0$ for all x (since the discriminant of x^2+2x+3 is negative), provided we take $a>0$. Therefore all such polynomials are of the form $P(x) = k^2(x^2-4x+C)$. ■

Alternative Solution. Writing $\Delta(x) = P(x^3+x) - P(x^2+1)$, we need $\Delta(x) \geq 0$ for all x; but we have $\Delta(1) = 0$, therefore needing $\Delta'(1) = 0$. Since $\Delta'(x) = (3x^2+1)P'(x^3+x) - 2xP'(x^2+1)$, this yields $P'(2) = 0$.

Writing $P(x) = ax^2+bx+c$ (with $a \neq 0$) means that $P'(2) = 0$ becomes $4a+b=0$ and so $-\frac{b}{a} = 4$, thus (if such a quadratic exists) the sum of its roots is $\boxed{4}$. Such polynomials in fact do exist (see the solution above). ■

Official Solution. The official solution is a model of the blind spot even a chess grandmaster, or an accomplished mathematician may occasionally have! Trying to avoid even these most rudimentary elements of continuity or differentiability for polynomials, featured in the solutions above, they so engaged in some extremely contrived argumentation, that I refuse to show here, in sheer disgust of it. ■

Problem 3. Point P is taken in the interior of the triangle ABC, so that

$$\angle PAB = \angle PCB = \frac{1}{4}(\angle A + \angle C).$$

Let L be the foot of the angle bisector of $\angle B$. The line PL meets the circumcircle of $\triangle APC$ at point Q. Prove QB is the angle bisector of $\angle AQC$.

(*S. Berlov*)

Solution. By some angle chasing, the figure is seen to be symmetric, with symmetry axis the angle bisector of $\angle B$. Then L belongs to the polar of point B with respect to the circumcircle of $\triangle APC$.

Let A_1 and C_1 be the second points of intersection of the circumcircle of $\triangle APC$ with CB and AB, respectively. Let S be the midpoint of the arc CA_1A and T be the midpoint of the arc A_1AC_1. Let Q_1 and Q_2 be the second points of intersection of the circumcircle of $\triangle APC$ with BS and BT, respectively. The point L is then the meeting point of the diagonals of the isosceles trapezium TSQ_1Q_2. We have now $\angle CQ_1S = \angle SQ_1A$ and $A_1T = TC_1$, thus $\angle TCB = \angle TAB = \dfrac{\angle A + \angle C}{4}$, so $T \equiv P$ and $Q_1 \equiv Q$. $\blacksquare$

Remarks. The configuration is quite pleasant, with lots of symmetry leading to other coincidences not mentioned in the statement; for example the fact that the circumcircle of $\triangle APC$ has as diameter II_B, the segment determined by the incentre and B-excentre of $\triangle ABC$.

The problem is not easy; it has only been solved by 3 out of the 35 competitors.

Problem 4. Let $p = 4k + 3$ be a prime. Prove that if

$$\frac{1}{0^2+1} + \frac{1}{1^2+1} + \cdots + \frac{1}{(p-1)^2+1} = \frac{m}{n},$$

where the fraction $\dfrac{m}{n}$ is in reduced terms, then $p \mid 2m - n$.

The mention of $\dfrac{m}{n}$ being in reduced terms is irrelevant; the conclusion is *a fortiori* true if m, n are not relatively prime.

(*A. Golovanov*)

Solution. Let us notice that p does not divide any of the denominators, since -1 is a non-quadratic residue modulo a prime $p \equiv 3 \pmod 4$. Denote the expression by E; having $E = \dfrac{m}{n}$ yields $2E - 1 = \dfrac{2m-n}{n}$, and the thesis now writes as needing to prove $2E - 1 \equiv 0 \pmod p$.

Consider $f(x) \in \mathbb{F}_p[x]$, $f(x) = \displaystyle\prod_{\ell=0}^{p-1}(x - \ell^2)$. Then, working in $\mathbb{F}_p[x]$,

$$f(x^2) = \prod_{\ell=0}^{p-1}(x^2 - \ell^2) = \left(\prod_{\ell=0}^{p-1}(x-\ell)\right)\left(\prod_{\ell=0}^{p-1}(x+\ell)\right) = (x^p - x)^2,$$

from Fermat's little theorem. The formal derivative of it is

$$(f(x^2))' = 2xf'(x^2) = 2(x^p - x)(px^{p-1} - 1) = -2(x^p - x).$$

But it is known (and easy to prove) that $\dfrac{f'(x)}{f(x)} = \displaystyle\sum_{\ell=0}^{p-1} \frac{1}{x - \ell^2}$, therefore

$$\sum_{\ell=0}^{p-1} \frac{1}{x^2 - \ell^2} = \frac{f'(x^2)}{f(x^2)} = \frac{-2(x^p - x)}{2x(x^p - x)^2} = -\frac{1}{x^2(x^{p-1} - 1)}.$$

Now comes the *coup de grâce*. In the field extension $\mathbb{F}_{p^2} = \mathbb{F}_p[x]/(x^2+1)$ there exists an element i such that $\mathrm{i}^2 = -1$. Taking $x = \mathrm{i}$ in the above expressions yields, on one hand, $\displaystyle\sum_{\ell=0}^{p-1} \frac{1}{\mathrm{i}^2 - \ell^2} = -\sum_{\ell=0}^{p-1} \frac{1}{1+\ell^2} = -E$, and on the other hand, $-\dfrac{1}{\mathrm{i}^2(\mathrm{i}^{p-1} - 1)} = -\dfrac{1}{2}$, since $\mathrm{i}^{p-1} = (\mathrm{i}^2)^{2k+1} = -1$. Therefore we get $E \equiv \dfrac{1}{2} \pmod p$, i.e. $2E - 1 \equiv 0 \pmod p$, as desired. ■

Alternative Solution. We may proceed in a slightly different way, which resembles more the solution for **Problem 4**, Junior League (which deals with a similar, but easier case). We will make use of the definition for i from above. Consider $g(x) \in \mathbb{F}_p[x]$, $g(x) = \displaystyle\prod_{\ell=0}^{p-1}(x - \ell) = x^p - x$, by Fermat's little theorem. We do have

$$\sum_{\ell=0}^{p-1} \frac{1}{x - \ell} = \frac{g'(x)}{g(x)} = \frac{px^{p-1} - 1}{x^p - x} = -\frac{1}{x(x^{p-1} - 1)}.$$

Computed at $x = \mathrm{i}$ it yields $-\dfrac{1}{\mathrm{i}(\mathrm{i}^{p-1} - 1)} = \dfrac{1}{2\mathrm{i}}$, since $\mathrm{i}^{p-1} = (\mathrm{i}^2)^{2k+1} = -1$. But now we can compute

$$E = \sum_{\ell=0}^{p-1} \frac{1}{\ell^2 + 1} = \sum_{\ell=0}^{p-1} \frac{1}{(\ell - \mathrm{i})(\ell + \mathrm{i})} = \frac{1}{2\mathrm{i}} \sum_{\ell=0}^{p-1} \left(\frac{1}{\ell - \mathrm{i}} - \frac{1}{\ell + \mathrm{i}} \right) = -\frac{1}{2\mathrm{i}} \sum_{\ell=0}^{p-1} \frac{2}{\mathrm{i} - \ell}.$$

By the computations made we thus have $E \equiv \left(-\dfrac{1}{\mathrm{i}}\right) \dfrac{1}{2\mathrm{i}} = \dfrac{1}{2} \pmod p$, and the proof is concluded as in the solution above. ■

Official Solution. Denote $a_\ell = \ell^2 + 1$. Then $\dfrac{m}{n} = \dfrac{\sigma_{p-1}(a_0, a_1, \ldots, a_{p-1})}{\sigma_p(a_0, a_1, \ldots, a_{p-1})}$, with $\sigma_j(a_0, a_1, \ldots, a_{p-1})$ the elementary symmetric polynomial of degree j. The polynomial of roots a_ℓ is $P(x) = \displaystyle\prod_{\ell=0}^{p-1}(x - 1 - \ell^2)$; with the substitution $t^2 = x-1$ we have $P(t^2+1) = \displaystyle\prod_{\ell=0}^{p-1}(t^2-\ell^2) = \left(\prod_{\ell=0}^{p-1}(t-\ell)\right)\left(\prod_{\ell=0}^{p-1}(t+\ell)\right)$, which seen in $\mathbb{F}_p[t]$ is $(t^p-t)^2 = t^{2p}-2t^{p+1}+t^2$. Performing the reverse substitution, $P(x) \equiv (x-1)^p-2(x-1)^{(p+1)/2}+(x-1) = x^p+\cdots+(p+2(p+1)/2+1)x-4$, thus by Viète's relation getting $\sigma_{p-1} \equiv 2 \pmod p$ and $\sigma_p \equiv 4 \pmod p$, therefore $p \mid 2\sigma_{p-1} - \sigma_p$. Since $p \nmid \sigma_p$, that is equivalent to $p \mid 2m - n$. ■

Alternative Solution. (Ömer Cerrahoğlu) The numbers from 2 to $p-2$ can be arranged in pairs of the form $\left(x, \dfrac{1}{x}\right) \pmod p$ (the numbers in each pair are different, as the only solutions to $x \equiv \dfrac{1}{x} \pmod p$ are 1 and -1). For any pair defined as above we have $\dfrac{1}{x^2+1} + \dfrac{1}{(1/x)^2+1} \equiv 1 \pmod p$ so we have that $\dfrac{1}{2^2+1} + \cdots + \dfrac{1}{(p-2)^2+1} \equiv \dfrac{p-3}{2} \pmod p$, so our sum is $\dfrac{p-3}{2} + 1 + 2 \cdot \dfrac{1}{2} = \dfrac{p+1}{2}$ modulo p, and we get the conclusion. ■

The problem is not easy if these techniques are not mastered (although Ömer's approach is quite direct and affordable, it may also be somewhat elusive); the problem has only been solved by 4 out of the 35 competitors. Strangely enough, a special (brilliancy) prize for an outstanding Number Theory solution was awarded for **Problem 5**, and to none of the few solvers of this **Problem 4** !

Problem 5. Solve in positive integer numbers the Diophantine equation

$$\frac{1}{n^2} - \frac{3}{2n^3} = \frac{1}{m^2}.$$

The requirement to solve the equation in **positive** integers does not make it any simpler, and is mostly irrelevant.

(*A. Golovanov*)

Solution. The equation writes $m^2(2n-3) = 2n^3$. Let $m = 2^a x$, $n = 2^b y$, with odd x, y. We then need have $2a = 3b + 1$, thus $a = 3t - 1$, $b = 2t - 1$, for some integer $t \geq 1$. The equation now writes $x^2(2^{2t}y - 3) = y^3$. But $\gcd(2^{2t}y - 3, y) \mid 3$.

• If $\gcd(2^{2t}y - 3, y) = 3$ we then need $y = 3z$, and so $x^2(2^{2t}z - 1) = 3^2 z^3$. But $\gcd(2^{2t}z - 1, z) = 1$, and so we need $2^{2t}z - 1 \mid 3^2$. There are only two potential cases

- $2^{2t}z - 1 = 3$, for $t = 1$ and $z = 1$, leading to $x^2 = 3$, impossible;
- $2^{2t}z - 1 = -9$, for $t = 1$ and $z = -2$, but z is odd, so impossible.

• If $\gcd(2^{2t}y - 3, y) = 1$ we then need $2^{2t}y - 3 = \pm 1$, hence $2^{2t}y = 4$. This leads to $t = 1$, $y = 1$, so $x = \pm 1$, therefore $\boxed{m = \pm 4 \text{ and } n = 2}$. ■

Alternative Solution. The equation writes $m^2 = \dfrac{2n^3}{2n-3}$, and again $(2m)^2 = \dfrac{8n^3}{2n-3} = (2n-3)^2 + 9(2n-3) + 27 + \dfrac{27}{2n-3}$. Thus one only has to check $2n - 3 \in \{-27, -9, -3, -1, 1, 3, 9, 27\}$ (we can quickly eliminate half of them, since clearly n must be even), and the only one that works is $2n - 3 = 1$, when $\boxed{n = 2 \text{ and } m = \pm 4}$. ■

Problem 6. Quadrilateral $ABCD$ is both cyclic and circumscribed. Its incircle touches its sides AB and CD at points X and Y, respectively. The perpendiculars to AB and CD drawn at A and D, respectively, meet at point U; those drawn at X and Y meet at point V, and finally, those drawn at B and C meet at point W. Prove that points U, V and W are collinear.

(*A. Golovanov*)

Solution. Let UV cut the perpendicular to CD through C at W'. Let $P \equiv AD \cap BC$. Wlog assume that the incircle (V) of $ABCD$ is P-excircle of $\triangle PAB$. The incircle (J) of $\triangle PAB$ touches AB at L. Since AB, DC are antiparallel with respect to PA, PB, then $\triangle PAB \sim \triangle PCD$ are similar, with incircles $(J), (V)$. Thus $\dfrac{UV}{VW'} = \dfrac{DY}{YC} = \dfrac{BL}{LA} = \dfrac{AX}{XB}$, having as a result $AU \parallel XV \parallel BW'$, hence $W \equiv W'$. ■

This very problem proved to be more tractable than **Problem 2**; more competitors managed their way through it.

Problem 7. Prove that for any real numbers a, b, c satisfying $abc = 1$ the following inequality holds

$$\frac{1}{2a^2+b^2+3} + \frac{1}{2b^2+c^2+3} + \frac{1}{2c^2+a^2+3} \leq \frac{1}{2}.$$

The original statement forced a, b, c to be positive; the comments following show why this is irrelevant.

(*V. Aksenov*)

Solution. We have $2a^2+b^2+3 = (a^2+b^2)+(a^2+1)+2 \geq 2|ab|+2|a|+2$ and similarly the other two cyclic ones, so

$$\sum_{cyc} \frac{1}{2a^2+b^2+3} \leq \frac{1}{2} \sum_{cyc} \frac{1}{|ab|+|a|+1}.$$

But $|a|(|bc|+|b|+1) = 1+|ab|+|a|$ and $|ab|(|ca|+|c|+1) = |a|+1+|ab|$, therefore

$$\frac{1}{2} \sum_{cyc} \frac{1}{|ab|+|a|+1} = \frac{1}{2(|ab|+|a|+1)} (1+|a|+|ab|) = \frac{1}{2}.$$

The equality cases obviously occur when $|a| = |b| = |c| = 1$ with $abc = 1$, thus for $(a, b, c) \in \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}$. ■

Remarks. Problem 1 of the 4$^{\text{th}}$ 2012 Romanian Junior Selection Test asked to prove the inequality

$$\sum_{cyc} \frac{1}{1+a^2+(b+1)^2} \leq \frac{1}{2}$$

for positive real numbers a, b, c satisfying $abc = 1$. Clearly the very same idea applies. The identity appearing at the end of the solution(s) has also been used, in a more complicated way, in Problem 1 of the 2011 Junior Stars of Mathematics competition.

More worrisome is that on July 13, 2012, the following has been posted on `mathlinks.ro` immediately followed by a solution (identical with the above)

Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$. Prove that

$$\frac{1}{a^2 + 2b^2 + 3} + \frac{1}{b^2 + 2c^2 + 3} + \frac{1}{c^2 + 2a^2 + 3} \leq \frac{1}{2}.$$

Of course, it is irrelevant to require a, b, c positive; we work with absolute values, the inequalities hold the same way, and only slightly more equality cases are possible. If one was fiendish, one could obscure even more the idea of the solution, by substituting $x = a^2$, $y = b^2$, $z = c^2$; we still have $xyz = 1$, but now the AM-GM manipulation of the denominators is even deeper hidden.

Problem 8. Integers not divisible by 2012 are arranged on the arcs of an oriented graph. We call the *weight of a vertex* the difference between the sum of the numbers on the arcs coming into it and the sum of the numbers on the arcs going away from it. It is known that the weight of each vertex is divisible by 2012. Prove that non-zero integers with absolute values not exceeding 2012 can be arranged on the arcs of this graph, so that the weight of each vertex is zero.

(*W. Tutte*)

Solution. The official solution is quite lengthy and complicated; the thesis is a particular case of a major theorem by Tutte (featured in [R. Diestel - *Graph Theory*]); it is not my intent to make here a presentation of it. ■

Only 2 among the 35 participants managed to earn one point (from 7); the rest ending up *tabula rasa*. Ahh, the dangers of featuring a deep theoretical result in competition ...

Problem 1. Tanya and Serezha take turns putting chips in empty squares of a chessboard. Tanya starts with a chip in an arbitrary square. At every next move, Serezha must put a chip in the column where Tanya put her last chip, while Tanya must put a chip in the row where Serezha put his last chip. The player who cannot make a move loses. Whom of the players has a winning strategy?

(*A. Golovanov*)

Solution. See **Problem 1**, Senior League.[4] ■

Problem 2. A rectangle $ABCD$ is given. Point K lies on the half-line $(DC$ so that $DK = DB$, and M is the midpoint of BK. Prove that AM is the angle bisector of $\angle BAC$.

(*S. Berlov*)

Solution. Clearly, DM is the angle bisector of $\angle BDC$, and $DM \perp BK$, so $ABMD$ is cyclic (its circumscribed circle is that of the rectangle), and $\angle BAM = \angle BDM = \dfrac{\angle BDC}{2} = \dfrac{\angle BAC}{2}$ (M is the midpoint of arc BC). ■

The problem proved to be quite easy; it has been solved by the vast majority of the 31 competitors.

Problem 3. Prove that N^2 arbitrary distinct positive integers $(N > 10)$ can be arranged in a $N \times N$ table, so that all $2N$ sums in rows and columns are distinct.

(*S. Volchenkov*)

Solution. The official solution goes by induction on $N \geq 2$, and runs for a considerable length (so why the $N > 10$ constraint in its statement?).

I will not yet present a solution, since I'm still trying to make something more palatable out of it. ■

The problem proved to be extremely difficult for the contestants; only 8 out of the 31 competitors managed as many as 3 points (from 7) (one 3, two 2 and five 1).

[4] Also see the commentary on the effect of not having specified a general $n \times n$ board!

Problem 4. Let be given the prime number $p = 1601$. Prove that if

$$\frac{1}{0^2+1} + \frac{1}{1^2+1} + \cdots + \frac{1}{(p-1)^2+1} = \frac{m}{n},$$

where we only sum over terms with denominators not divisible by p, and the fraction $\frac{m}{n}$ is in reduced terms, then $p \mid 2m + n$.

The mention of $\frac{m}{n}$ being in reduced terms is irrelevant; the conclusion is *a fortiori* true if m, n are not co-prime. Patronizing to state 1601 is a prime.

(*A. Golovanov*)

Solution. Similar with **Problem 4**, Senior League; except that here $p = 4k + 1$ and $p \mid 2m + n$ (it is essential $p = 1601$ is a prime of this form). However, the proceedings are much simplified by the fact that now -1 is a quadratic residue modulo p, namely that there exist two elements $\pm\mathrm{i} \in \mathbb{F}_p$ such that $\mathrm{i}^2 = -1$. By Wilson's theorem, we may even exhibit these two special elements. We have $(p-1)! \equiv -1 \pmod p$, but also

$$(p-1)! = \left(\prod_{\ell=1}^{(p-1)/2} \ell\right)\left(\prod_{\ell=1}^{(p-1)/2} (p-\ell)\right) \equiv (-1)^{\frac{p-1}{2}}(((p-1)/2)!)^2 \pmod p,$$

so $\mathrm{i} = ((p-1)/2)!$. In the case at hand, it is immediately visible that since $p = 1601 = 40^2 + 1$ we will have $\mathrm{i} = 40$ (although knowing the exact value is irrelevant in the sequel).

Let us notice now that p does not divide any of the denominators $\ell^2 + 1$ for $\ell \neq \pm\mathrm{i}$ (and it divides $\mathrm{i}^2 + 1$ and $(-\mathrm{i})^2 + 1$). Denote the expression by E, so $E = \displaystyle\sum_{\ell \neq \pm\mathrm{i}} \frac{1}{\ell^2+1}$; having $E = \dfrac{m}{n}$ yields $2E + 1 = \dfrac{2m+n}{n}$, and the thesis now writes as needing to prove $2E + 1 \equiv 0 \pmod p$. But

$$E = \sum_{\ell \neq \pm\mathrm{i}} \frac{1}{\ell^2+1} = \sum_{\ell \neq \pm\mathrm{i}} \frac{1}{(\ell-\mathrm{i})(\ell+\mathrm{i})} = \frac{1}{2\mathrm{i}} \sum_{\ell \neq \pm\mathrm{i}} \left(\frac{1}{\ell-\mathrm{i}} - \frac{1}{\ell+\mathrm{i}}\right).$$

The sum above telescopes to $\dfrac{1}{2\mathrm{i}}\left(\dfrac{1}{2\mathrm{i}} - \dfrac{1}{-2\mathrm{i}}\right) = \dfrac{1}{2\mathrm{i}^2} = -\dfrac{1}{2}$. Therefore we get $E \equiv -\dfrac{1}{2} \pmod p$, i.e. $2E + 1 \equiv 0 \pmod p$, as desired. ■

Only 6 among the 31 participants managed to earn one point (from 7); the rest ending up *tabula rasa*. Perceived as more difficult than actually was.

Problem 5. The vertices of a regular polygon with 2012 sides are labeled $A_1, A_2, \ldots, A_{2012}$ in some order. It is known that if $k+\ell$ and $m+n$ leave the same remainder when divided by 2012, then the chords A_kA_ℓ and A_mA_n have no common points. Vasya walks around the polygon and sees that the first two vertices are labeled A_1 and A_4. How is the tenth vertex labeled?

(*A. Golovanov*)

Solution. Consider three consecutive vertices labeled A_k, A_m, A_ℓ. Take the unique $1 \leq n \leq 2012$ for which $n \equiv (k+\ell) - m \pmod{2012}$. Since then the chords A_kA_ℓ and A_mA_n would meet, it means the only possibility is to have $n = m$; this forces $k+\ell$ to be even, and $\ell \equiv 2m - k \pmod{2012}$. This can also be interpreted as having the labels following an arithmetic progression, of ratio $m - k \pmod{2012}$

Applied to our case, with $k = 1$ and $m = 4$, this yields the t-th vertex to be labeled $A_{1+(3(t-1) \pmod{2012})}$, hence the tenth label will be $\boxed{A_{28}}$.

It behooves a true mathematician (even if not asked in the problem) to assure oneself that this labeling actually obeys the conditions stated. First, let us notice that since $3 \nmid 2012$, the values $1 + (3(t-1) \pmod{2012})$ run over all possible indices $\{1, 2, \ldots, 2012\}$ of the labels when $1 \leq t \leq 2012$.

Now, for two vertices labeled with indices $1 + (3(k-1) \pmod{2012})$ and $1 + (3(\ell-1) \pmod{2012})$, for $1 \leq k < \ell \leq 2012$, then to the vertex labeled with index $1 + (3(m-1) \pmod{2012})$, for some $k < m < \ell$, corresponds a vertex labeled with index $1 + (3(n-1) \pmod{2012})$ where $n = (k+\ell) - m$, thus also $k < n < \ell$, and clearly the corresponding chords do not meet. Therefore the labeling is consistent with the conditions of the problem; the only thing that matters is that $3 \nmid 2012$. ■

Problem 6. Solve in positive integer numbers the Diophantine equation

$$\frac{1}{n^2} - \frac{3}{2n^3} = \frac{1}{m^2}.$$

The requirement to solve the equation in **positive** integers does not make it any simpler, and is mostly irrelevant.

(*A. Golovanov*)

Solution. See **Problem 5**, Senior League. ■

Problem 7. A circle is contained in a quadrilateral with successive sides of lengths $3, 6, 5$ and 8. Prove that the length of its radius is less than 3.

(*K. Kokhas*)

Solution. Although I am not usually delving into geometry problems, I will take exception this time, since probably another unexpected turn may have occurred in what the problem setters were concerned; they failed to consider an argument of areas. Clearly if the area K of the quadrilateral is lesser than 9π, any contained circle will have radius of length lesser than 3.

The fact that the area of a quadrilateral is

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta},$$

where a, b, c, d are the lengths of the sides of the quadrilateral, s is its semi-perimeter, and θ is the semi-sum of two opposite angles, is known as Bretschneider's formula. Thus $K \leq \sqrt{(s-a)(s-b)(s-c)(s-d)}$, with equality when the quadrilateral is cyclic; this is Brahmagupta's formula. So, in our case, $K \leq \sqrt{(11-3)(11-5)(11-6)(11-8)} = 3\sqrt{80} < 27 < 9\pi$. Another argument of area could potentially be based on Ptolemy's inequality. If δ_1, δ_2 are the lengths of the diagonals and φ the angle made by them, then, if the successive side lengths were $3, 6, 8, 5$, we would have

$$K = \frac{\delta_1\delta_2 \sin\varphi}{2} \leq \frac{\delta_1\delta_2}{2} \leq \frac{3\cdot 8 + 5\cdot 6}{2} = 27 < 9\pi,$$

which again yields the thesis. But at least the problem setters avoided this pitfall; as it is (with successive side lengths $3, 6, 5, 8$), the above approach gives

$$K = \frac{\delta_1\delta_2 \sin\varphi}{2} \leq \frac{\delta_1\delta_2}{2} \leq \frac{3\cdot 5 + 6\cdot 8}{2} = \frac{63}{2} > 9\pi.$$

No matter; to each measure – a counter-measure! By denoting the angles between the sides of lengths $8, 3$ and $6, 5$ be α, respectively β, then

$$K = \frac{8\cdot 3\cdot \sin\alpha + 6\cdot 5\cdot \sin\beta}{2} \leq \frac{8\cdot 3 + 6\cdot 5}{2} = 27 < 9\pi.$$

Area arguments kill the problem; both official solutions are contrived. ■

The problem was not perceived as easy; it has only been solved by 4 out of the 31 competitors.

Problem 8. 25 little donkeys stand in a row; the rightmost of them is Eeyore.[5] Winnie-the-Pooh wants to give a balloon of one of the seven colours of the rainbow to each donkey, so that successive donkeys receive balloons of different colours, and so that at least one balloon of each colour is given to some donkey. Eeyore wants to give to each of the 24 remaining donkeys a pot of one of six colours of the rainbow (except red), so that at least one pot of each colour is given to some donkey (but successive donkeys **can** receive pots of the same colour). Whom of the two friends has more ways to get his plan implemented, and how many times more?

(*F. Petrov*)

Solution. In fact both these numbers can be computed. Label the colours of the rainbow by $1, 2, 3, 4, 5, 6, 7$, with red being the 7-th. Denote by A_k, $1 \leq k \leq 7$, the set of ways to give balloons of any but the k-th colour to 25 donkeys, so that neighbouring donkeys receive balloons of different colour (though maybe not all 6 of the available colours are being used). Clearly, for any $K \subseteq \{1, 2, 3, 4, 5, 6, 7\}$, we have $\left|\bigcap_{k\in K} A_k\right| = (7 - |K|)(6 - |K|)^{24}$. By the Principle of Inclusion/Exclusion (PIE) we have

$$\left|\bigcup_{k=1}^{7} A_k\right| = \sum_{\ell=1}^{7}(-1)^{\ell-1}\sum_{|K|=\ell}\left|\bigcap_{k\in K} A_k\right| = \sum_{m=0}^{6}(-1)^m\binom{7}{m}m(m-1)^{24}.$$

Now, denote by A_0 the set of ways to give balloons of any of the colours of the rainbow to 25 donkeys, so that neighbouring donkeys receive balloons of different colour (though maybe not all 7 of the available colours are being used). Clearly $|A_0| = 7 \cdot 6^{24}$. The number we are looking for, the number of such ways that use all 7 colors, is therefore

$$|A_0| - \left|\bigcup_{k=1}^{7} A_k\right| = \sum_{m=0}^{7}(-1)^{m-1}\binom{7}{m}m(m-1)^{24}.$$

We have now to compute the number of ways to distribute pots, i.e. 6 colours (no red), 24 donkeys, no restriction for neighbouring donkeys, but

[5] Eeyore is a character in the Winnie-the-Pooh books by A. A. Milne. He is generally depicted as a pessimistic, gloomy, depressed, old grey stuffed donkey, who is a friend of the title character, Winnie-the-Pooh. His name is an onomatopœic representation of the braying sound made by a normal donkey. Of course, Winnie-the-Pooh is a fictional anthropomorphic bear.

all 6 colours to be used. Denote, following the fashion of above, by B_k, $1 \le k \le 6$, the set of ways to give pots of any but red and the k-th colour to 24 donkeys (though maybe not all 5 of the available colours are being used). Clearly, for any $K \subseteq \{1,2,3,4,5,6\}$, we have $\left|\bigcap_{k\in K} B_k\right| = (6-|K|)^{24}$. By the Principle of Inclusion/Exclusion (PIE) we have

$$\left|\bigcup_{k=1}^{6} B_k\right| = \sum_{\ell=1}^{6}(-1)^{\ell-1}\sum_{|K|=\ell}\left|\bigcap_{k\in K} B_k\right| = \sum_{m=0}^{5}(-1)^{m-1}\binom{6}{m}m^{24}.$$

Now, denote by B_0 the set of ways to give pots of any of the colours of the rainbow but red to 24 donkeys (though maybe not all 6 of the available colours are being used). Clearly $|B_0| = 6^{24}$. The number we are looking for, the number of such ways that use all 6 colors, is therefore

$$|B_0| - \left|\bigcup_{k=1}^{6} B_k\right| = \sum_{m=0}^{6}(-1)^{m}\binom{6}{m}m^{24}.$$

All it remains is to notice that, term by term,

$$\sum_{m=1}^{7}(-1)^{m-1}\binom{7}{m}m(m-1)^{24} = 7\sum_{m=0}^{6}(-1)^{m}\binom{6}{m}m^{24},$$

so there are $\boxed{7}$ more ways to distribute balloons than pots. ■

Alternative Official Solution. This method does not actually count the ways, but rather establishes a one-to-one correspondence between Eeyore's ways to distribute pots and Winnie's ways to distribute balloons, all the while giving Eeyore a red one. In a nutshell, pots and balloons have the same color, except when a pair of two consecutive pots share a same color, in which case the colour of the second balloon in the pair is red.

Now, the total number of ways Winnie can distribute balloons is $\boxed{7}$ times larger, since the balloon given to Eeyore may be of any of the 7 colours, not just red, and clearly for each of the colours of the balloon given to Eeyore there are a same number of ways to do it. ■

The problem was perceived as difficult; it has only been solved by 2 out of the 31 competitors, the rest getting just zero points.

Afterword. Some of the solutions – mostly the geometry ones – have been culled from the Internet (`mathlinks.ro`); otherwise the vast amount of this article is the original work of the author.

Is it symptomatic that nobody else in Romania is at all interested in analyzing these problems of a competition where a national team represented the country, and neither in presenting a comprehensive set of solutions and commentaries?

Romania participated (on the mathematics panel of the competition) with three competitors, all in the Seniors League; **Florina Toma** (of Iaşi) earned one of the 4 gold medals (with 32/56 points), while **Andreea Măgălie** (ICHB) and **Ştefan Gramatovici** (T. Vianu College) each earned one of the 7 silver medals (with 28/56 points). Uncharacteristic oversights (blame it on the faraway venue of the competition!) deprived them of earning gold, which surely was within their reach. The T. Vianu College also sent a (private) team of three (two in the Seniors League, one in the Juniors), without noteworthy achievements.

DAN SCHWARZ — July 22, 2012