Test de Departajare pentru MofM 2014 (Bucureşti) – Enunţuri & Soluţii

Problem 1. Given $n+1$ distinct real numbers in the interval $[0,1]$, prove there exist two of them $a \neq b$, such that $ab|a-b| < \dfrac{1}{3n}$.

AoPS

Solution. Index the numbers $0 \le a_0 < a_1 < \cdots < a_n \le 1$. If $a_0 = 0$ we're done; if not, $\sum_{k=0}^{n-1} a_{k+1}a_k(a_{k+1}-a_k) = \dfrac{1}{3}\left(a_n^3 - a_0^3 - \sum_{k=0}^{n-1}(a_{k+1}-a_k)^3\right) < \dfrac{1}{3}$, so there will exist $0 \le k \le n-1$ such that $\boxed{a_{k+1}a_k(a_{k+1}-a_k) < \dfrac{1}{3n}}$ (by an averaging argument). ■

Alternative Solution. (L. Ploscaru) Indexăm numerele $0 \le a_0 < a_1 < \cdots < a_n \le 1$. Din principiul cutiei, va exista un indice $0 \le k \le n-1$ astfel încât $a_{k+1}^3 - a_k^3 \le \dfrac{1}{n}$. Dar avem $a_{k+1}^3 - a_k^3 = (a_{k+1}-a_k)((a_{k+1}-a_k)^2 + 3a_{k+1}a_k) > 3a_{k+1}a_k(a_{k+1}-a_k)$ (cu inegalitate strictă, căci $a_{k+1} \neq a_k$), deci $3a_{k+1}a_k(a_{k+1}-a_k) < \dfrac{1}{n}$. ■

Remark. The main idea is to notice that cubes of the variables are of good use, via the equivalent forms $3ab(a-b) = a^3 - b^3 - (a-b)^3$ or $a^3 - b^3 = (a-b)((a-b)^2 + 3ab)$.

Problem 2. What is the minimum number $m(n)$ of edges of K_n (the complete graph on $n \ge 4$ vertices) that can be colored red, such that any K_4 subgraph contains a red K_3? For example, $m(4) = 3$.

AoPS

Solution. The answer is in fact quite easy to get. Assume the edge ab is not red. Then the fact that among any $\{a,b,x,y\}$ has to exist a red triangle forces xy to be red, and moreover, either ax, ay to be red or bx, by to be red. That means $K_n - \{a,b\} = K_{n-2}$ is red. Let A be the set of vertices x such that ax is red, and B be the set of vertices y such that by is red; it follows $A \cup B = K_n \setminus \{a,b\}$. If we could take $x \in A \setminus B$ and $y \in B \setminus A$, then $\{a,b,x,y\}$ would be a contradiction, so say $B \setminus A = \emptyset$, thus $A = K_n \setminus \{a,b\}$, therefore $K_n - \{b\} = K_{n-1}$ is red. That is enough, so $\boxed{m(n) = (n-1)(n-2)/2}$. ■

Alternative Solution. Consider the largest red clique K_r contained in such a graph G. If $r \le n-2$, for any $v \in G - K_r$ there must exist some $f(v) \in K_r$ such that the edge $vf(v)$ is not red. Now, if there exist $v, w \in G - K_r$ such that $f(v) \neq f(w)$, then the subgraph induced by $\{v, w, f(v), f(w)\}$ cannot contain a red triangle. On the other hand, having $f(v) = \omega$ constant for all $v \in G - K_r$, makes that in order for the subgraph induced by $\{v, w, \omega, u\}$, for arbitrary $v, w \in G - K_r$ and $u \in G - \omega$, to contain a red triangle, we need vw to be red; but then $K_{n-1} = G - \omega$ is a red clique. ■

Alternative Solution. (A. Măgălie) Demonstrăm că în orice astfel de graf va exista (măcar) un vârf de grad roşu cel puţin $n-2$. Cum $m(4) = 3$, rezultă prin inducţie că $m(n+1) \ge m(n) + (n-1) = (n-1)(n-2)/2 + (n-1) = n(n-1)/2$, cu modelul minim descris mai sus. ■

Problem 3. Let $0 < p \le Q$ be fixed real numbers, and let a, b, x and y be positive real numbers, such that $\begin{cases} ax & \le p \\ ay & \le Q \\ bx & \le Q \\ by & \le Q \end{cases}$. Determine the maximum value of $(a+b)(x+y)$, and the cases of equality.

Sgall's Lemma

Solution. Let us normalize, by taking $\lambda = \dfrac{y}{x}$, $\mu = \dfrac{b}{a}$, $m = \min\{\lambda, \mu\}$, $M = \max\{\lambda, \mu\}$, $p' = \dfrac{p}{ax}$ and $Q' = \dfrac{Q}{ax}$, and dividing all inequations by ax, to get $\begin{cases} 1 & \le p' \\ m & \le Q' \\ M & \le Q' \\ mM & \le Q' \end{cases}$.

We thus need to maximize $(1+m)(1+M)$. We claim the maximum is $2(p'+Q')$.

• If $m < 1$, then $1+m+M+mM < 2+2M \le 2+2Q' \le 2(p'+Q')$.
• If $1 \le m$, then $(m-1)(M-1) \ge 0$, so $m+M \le 1+mM$, thus $1+m+M+mM \le 2(1+mM) \le 2(p'+Q')$. Equality is reached if and only if $p' = 1$, $Q' = M$ and $m = 1$.

Going back to the original variables, the above means $\boxed{(a+b)(x+y) \le 2(p+Q)}$, with equality occuring if and only if $\boxed{p = ax,\, Q = by}$ and $\boxed{y = x \text{ or } b = a}$. ■

Problem 4. Say that a (nondegenerate) triangle is *funny* if it satisfies the condition that the altitude, median, and angle bisector drawn from one of the vertices partition the triangle into 4 non-overlapping triangles whose areas form (in some order) a 4-term arithmetic sequence. (One of these 4 triangles is allowed to be degenerate.) Find, with proof, all funny triangles.

MATH PRIZE FOR GIRLS 2013

Solution. (L. Ploscaru) Să presupunem că cele trei ceviene pleacă din A, cu $AB < AC$ ($\triangle ABC$ nu poate evident fi isoscel în A; din ipoteză se deduce şi că triunghiul *funny* nu poate fi obtuzunghic în B sau C.). Ordinea dreptelor este

$$AB - \text{înălţimea} - \text{bisectoarea} - \text{mediana} - AC$$

(se demonstrează eventual uitându-ne la picioarele lor pe BC). Ideea principală este să demonstrăm că un triunghi *funny* ABC e dreptunghic (paranteza din ipoteză face aluzie la această posibilitate; dacă nu erau triunghiuri *funny* dreptunghice, nu îşi avea rostul).

Să zicem că M este mijlocul lui BC; atunci aria$[ABM]$ = aria$[ACM]$, deci clar ACM e triunghiul cu cea mai mare arie. Fie q, $q+r$, $q+2r$, $q+3r$ ariile. Cele 3 triunghiuri mici îl partiţionează pe ABM, deci $3q+3r = q+3r$, de unde $q = 0$, iar atunci singurul fel în care 2 din cele 5 drepte de mai sus pot coincide este $AB \perp BC$, adică $\triangle ABC$ este dreptunghic în B (în afară de cazul imposibil când $\triangle ABC$ este isoscel în A).

Acum problema e aproape gata; luăm D piciorul bisectoarei, şi prin simpla formulă aria $= \dfrac{1}{2}$baza$\times$înălţimea, vom obţine că $\{BD, DM, MC\} = \{x, 2x, 3x\}$ pentru un x real pozitiv. Evident $MC = 3x$, iar atunci în fiecare dintre cele două cazuri aplicăm teorema bisectoarei ca să aflăm valoarea raportului $AB/AC = \cos A$, şi am terminat. Obţinem $\angle A \in \{\arccos(1/5), \arccos(1/2) = \pi/3\}$ (deci unul dintre triunghiuri este cel de unghiuri $30°, 60°, 90°$, dar mai există un caz). ■

Problem 5. For positive real numbers a, b, c with $a^2+b^2+c^2 \ge 3$, prove the inequality

$$\frac{a^2}{1+bc} + \frac{b^2}{1+ca} + \frac{c^2}{1+ab} \ge \frac{3}{2}$$

and determine its case(s) of equality.

Show that if $a^2+b^2+c^2 < 3$, the inequality may hold no more.

DAN SCHWARZ, variant of Italian Test

Solution. It is enough to consider the case $a^2+b^2+c^2 = 3$. Indeed, for $k \ge 1$ we have $\dfrac{(ka)^2}{1+(kb)(kc)} \ge \dfrac{a^2}{1+bc}$ *et.al.*

We then have $1 + bc \leq 1 + \dfrac{b^2 + c^2}{2} = \dfrac{5 - a^2}{2}$, hence $\dfrac{a^2}{1+bc} \geq \dfrac{2a^2}{5-a^2}$ *et.al.* Then the function $f\colon [0,3] \to \mathbb{R}$ given by $f(t) = \dfrac{2t}{5-t} = \dfrac{10}{5-t} - 2$ is clearly convex, therefore we have (by Jensen's inequality)

$$f(a^2) + f(b^2) + f(c^2) \geq 3f\left(\frac{a^2+b^2+c^2}{3}\right) = 3f(1) = \frac{3}{2}.$$

Thus the inequality is proved, with the obvious equality case when $a^2 + b^2 + c^2 = 3$ and $a = b = c = 1$.

For $a^2+b^2+c^2 < 3$ the inequality will hold no more; just consider $0 < a = b = c = k < 1$, and then LHS $= \dfrac{3k^2}{1+k^2} < \dfrac{3}{2}$. ■

Alternative Solution. Trying the Cauchy-Schwarz inequality, just for $a^2 + b^2 + c^2 = 3$ (seen to be enough)

$$\frac{a^2}{1+bc} + \frac{b^2}{1+ca} + \frac{c^2}{1+ab} \geq \frac{(a+b+c)^2}{3+bc+ca+ab} = \frac{3+2(ab+bc+ca)}{3+ab+bc+ca}$$

will not work this time, since the hopeful continuation towards value $\dfrac{3}{2}$ would require $6 + 4(ab+bc+ca) \geq 9 + 3(ab+bc+ca)$, *i.e.* $ab+bc+ca \geq 3$, which in fact it is precisely the other way around.

If however we try a common trick, and write

$$\frac{a^2}{1+bc} + \frac{b^2}{1+ca} + \frac{c^2}{1+ab} = \frac{a^4}{a^2+a^2bc} + \frac{b^4}{b^2+b^2ca} + \frac{c^4}{c^2+c^2ab},$$

then we can continue by Cauchy-Schwarz

$$\frac{a^4}{a^2+a^2bc} + \frac{b^4}{b^2+b^2ca} + \frac{c^4}{c^2+c^2ab} \geq \frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)+abc(a+b+c)} = \frac{9}{3+abc(a+b+c)}.$$

Now, in order to continue with $\geq \dfrac{3}{2}$, we need $abc(a+b+c) \leq 3$, which holds true, since $abc \leq \left(\dfrac{a^2+b^2+c^2}{3}\right)^{3/2} = 1$ and $a+b+c \leq \sqrt{3(a^2+b^2+c^2)} = 3$; the equality case follows as above. ■

Alternative Solution. (C. Popescu) The required inequality is a consequence of the following inequality

$$\sum \frac{a^2}{1+bc} \geq \frac{3(a^2+b^2+c^2)}{3+a^2+b^2+c^2}.$$

To prove the latter, apply Jensen's inequality to the convex function $t \mapsto (1+t)^{-1}$, $t > -1$, at $t_1 = bc$, $t_2 = ca$ and $t_3 = ab$, with weights $\lambda_1 = a^2/(a^2+b^2+c^2)$, $\lambda_2 = b^2/(a^2+b^2+c^2)$ and $\lambda_3 = c^2/(a^2+b^2+c^2)$, respectively, to obtain

$$\sum \frac{a^2}{a^2+b^2+c^2} \cdot \frac{1}{1+bc} \geq \frac{1}{1+\sum \frac{a^2}{a^2+b^2+c^2} \cdot bc} = \frac{a^2+b^2+c^2}{a^2+b^2+c^2+abc(a+b+c)},$$

and get thereby

$$\sum \frac{a^2}{1+bc} \geq \frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+abc(a+b+c)}.$$

Now,

$$abc(a+b+c) \leq \frac{1}{3^3}(a+b+c)^3(a+b+c) \leq \frac{1}{3^3} \cdot 3^2(a^2+b^2+c^2)^2 = \frac{1}{3}(a^2+b^2+c^2)^2,$$

so

$$\sum \frac{a^2}{1+bc} \geq \frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\frac{1}{3}(a^2+b^2+c^2)^2} = \frac{3(a^2+b^2+c^2)}{3+a^2+b^2+c^2}.$$

This ends the proof. ■

Remarks. Notice that the following again works immediately.

$$\frac{a^4}{1+bc}+\frac{b^4}{1+ca}+\frac{c^4}{1+ab}\geq\frac{(a^2+b^2+c^2)^2}{3+bc+ca+ab}=\frac{9}{3+ab+bc+ca}\geq\frac{3}{2}.$$

The original Italian Test problem was to prove for $a^2+b^2+c^2=3$ the inequality

$$\frac{1}{1+bc}+\frac{1}{1+ca}+\frac{1}{1+ab}\geq\frac{3}{2},$$

much easier to handle. A "brute force" solution is also possible here, but more difficult to compute for the variant asked above. In fact $a^2+b^2+c^2\leq 3$ is both needed, and enough, for the Italian problem.

Combining the two, both holding for $a^2+b^2+c^2=3$, allows us to then claim that

$$\frac{1+a^2}{1+bc}+\frac{1+b^2}{1+ca}+\frac{1+c^2}{1+ab}\geq 3.$$

Problem 6. Find the formula of the general term of a real numbers sequence $(x_n)_{n\geq 1}$ satisfying

$$\begin{cases} x_1=3 \\ 3(x_{n+1}-x_n)=\sqrt{x_{n+1}^2+16}+\sqrt{x_n^2+16} \end{cases}$$

AoPS

Solution. It is clear the sequence is (strictly) increasing. Then

$$3(x_{n+1}-x_n)=\sqrt{x_{n+1}^2+16}+\sqrt{x_n^2+16}=\frac{(x_{n+1}-x_n)(x_{n+1}+x_n)}{\sqrt{x_{n+1}^2+16}-\sqrt{x_n^2+16}}$$

allows us to write $x_{n+1}+x_n=3\left(\sqrt{x_{n+1}^2+16}-\sqrt{x_n^2+16}\right)$. So $4x_{n+1}-5x_n=3\sqrt{x_n^2+16}$. Square it, write it for the next index, subtract the two and factorize, in order to get $8(x_{n+2}-x_n)(2x_{n+2}-5x_{n+1}+2x_n)=0$, hence $2x_{n+2}-5x_{n+1}+2x_n=0$. By the known methods, the general solution is $x_n=\alpha 2^n+\beta 2^{-n}$. Since the sequence can in fact be prolonged to the left, to $x_0=0$, the coefficients can be determined to be $\alpha=2$, $\beta=-2$, so $\boxed{x_n=2^{n+1}-2^{-n+1}}$. ■

Alternative Solution. If we compute the first few terms and "guess" this formula, it is a simple task to check it verifies the recurrence relation, since

$$3(x_{n+1}-x_n)=3\left(2^{n+2}-\frac{1}{2^n}-2^{n+1}+\frac{1}{2^{n-1}}\right)=3\left(2^{n+1}+\frac{1}{2^n}\right),$$

$$\sqrt{x_{n+1}^2+16}+\sqrt{x_n^2+16}=\left(2^{n+2}+\frac{1}{2^n}\right)+\left(2^{n+1}+\frac{1}{2^{n-1}}\right)=3\left(2^{n+1}+\frac{1}{2^n}\right).$$

The fact that the sequence is uniquely determined by its first term immediately follows from the fact that the mapping $x\mapsto 3x-\sqrt{x^2+16}$ is increasing from $-\infty$ to $+\infty$, thus one-to-one, or from the previously obtained relation $4x_{n+1}=5x_n+3\sqrt{x_n^2+16}$. There are merits in this approach, especially if one has seen in the past such relations. ■

Alternative Solution. (M. Ivan) Se vede că şirul este strict crescător, deci cu termeni pozitivi. Deoarece funcţia $f\colon(1,\infty)\to(0,\infty)$ dată de $f(x)=2\left(x-\frac{1}{x}\right)$ este bijectivă, putem scrie unic $x_n=2\left(a_n-\frac{1}{a_n}\right)$, cu $a_1=2$ şi $a_n>1$ pentru toţi $n>1$. Relaţia de recurenţă conduce atunci la $a_{n+1}=2a_n$, deci $a_n=2^n$ pentru orice $n\geq 1$. Prin urmare $x_n=2\left(2^n-\frac{1}{2^n}\right)$, care evident verifică. ■

Sfârşit. Soluţii originale şi compilate de Dan Schwarz, 9 decembrie 2013

More developments on Problem 1. Given $n+1$ distinct real numbers in the interval $[0,1]$, prove there exist two of them $a \neq b$, such that $ab|a-b| < \dfrac{1}{3n}$.

Solution. Index the numbers $0 \le a_0 < a_1 < \cdots < a_n \le 1$. If $a_0 = 0$ we're done; if not, $\sum_{k=0}^{n-1} a_{k+1}a_k(a_{k+1}-a_k) = \dfrac{1}{3}\left(a_n^3 - a_0^3 - \sum_{k=0}^{n-1}(a_{k+1}-a_k)^3\right) < \dfrac{1}{3}$, so there will exist $0 \le k \le n-1$ such that $\boxed{a_{k+1}a_k(a_{k+1}-a_k) < \dfrac{1}{3n}}$ (by an averaging argument).

We can say more. We have $\sum_{k=0}^{n-1}(a_{k+1}-a_k)^3 \ge \dfrac{1}{n^2}\left(\sum_{k=0}^{n-1}(a_{k+1}-a_k)\right)^3 = \dfrac{1}{n^2}(a_n-a_0)^3$, by Hölder's inequality, so the bound becomes $\dfrac{1}{3}\left(a_n^3 - a_0^3 - \dfrac{1}{n^2}(a_n-a_0)^3\right) \le \dfrac{1}{3}\left(1 - \dfrac{1}{(n+1)^2}\right)$, by the following

LEMMA 1. For $0 \le x < y \le 1$, the maximum of $y^3 - x^3 - \dfrac{1}{n^2}(y-x)^3$ is $1 - \dfrac{1}{(n+1)^2}$.

Proof. Let $f(x,y) = y^3 - x^3 - \dfrac{1}{n^2}(y-x)^3$; then $\dfrac{\mathrm{d}}{\mathrm{d}y} f(x,y) = 3y^2 - \dfrac{3}{n^2}(y-x)^2$ vanishes at $(n-1)y = -x$, outside $(0,1]$, so $\max_{x<y\le 1} f(x,y) = f(x,1) = 1 - x^3 - \dfrac{1}{n^2}(1-x)^3$. Accordingly, $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x,1) = -3x^2 + \dfrac{3}{n^2}(1-x)^2$ vanishes at $x = \dfrac{1}{n+1}$, so $\max_{0\le x<1} f(x,1) = 1 - \dfrac{1}{(n+1)^2}$. □

The equality case in Hölder forces $a_k = a_0 + \dfrac{k}{n+1}$, thus with $a_0 = \dfrac{1}{n+1}$ it yields $a_k = \dfrac{k+1}{n+1}$ (and indeed $a_n = \dfrac{n+1}{n+1} = 1$). But then still the summands are not all equal, so $1 - \dfrac{1}{(n+1)^2}$ is not reachable. For that we would need all summands equal, and that possibility has to be explored. However, we still improved – to the existence of a, b with

$$ab|a-b| < \frac{1}{3n}\left(1 - \frac{1}{(n+1)^2}\right).$$

A second course of action is to explore the maximum of $y^3 - x^3 - (y-x)^3 = 3yx(y-x)$, with $\dfrac{k}{n} \le x^3 < y^3 \le \dfrac{k+1}{n}$, and apply it to the Alternate Solution.

LEMMA 2. For $0 \le a \le x^3 < y^3 \le b \le 1$, the maximum of $y^3 - x^3 - (y-x)^3 = 3yx(y-x)$ is as shown below.

Proof. Seen as a function in y, it is a parabola with apex at $y = x/2$, thus having it maximum at $y^3 = b$. Seen now as a function in x, it is a parabola with both apex and maximum at $x = \sqrt[3]{b}/2$, namely the maximum will be $3b/4$ if $\sqrt[3]{b}/2 \ge \sqrt[3]{a}$, but otherwise $3\sqrt[3]{ab}(\sqrt[3]{b} - \sqrt[3]{a})$. □

A third course of action is to maximize a common value for each term (using $a_n = 1$).

Alternative Approach. (M. Bălună) Putem vedea expresia $\sum_{k=0}^{n-1} a_{k+1}a_k(a_{k+1}-a_k)$, cu o minimă transformare, drept sumă Riemann sau Darboux. Avem, pentru $0 \le a < b \le 1$, $ba < \dfrac{1}{4}(b+a)^2$, şi $\dfrac{1}{4}(b+a)^2(b-a) < \dfrac{1}{3}(b^3-a^3) = \int_a^b x^2\,\mathrm{d}x$, căci ambele revin la $(b-a)^2 > 0$. Dar atunci $\sum_{k=0}^{n-1} a_{k+1}a_k(a_{k+1}-a_k) < \sum_{k=0}^{n-1} \dfrac{1}{4}(a_{k+1}+a_k)^2(a_{k+1}-a_k) < \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} x^2\,\mathrm{d}x$, şi cum $\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} x^2\,\mathrm{d}x = \int_{a_0}^{a_n} x^2\,\mathrm{d}x \le \int_0^1 x^2\,\mathrm{d}x = \dfrac{1}{3}$, vom găsi un k cu $a_{k+1}a_k(a_{k+1}-a_k) < \dfrac{1}{3n}$.