SOLUŢII ŞI COMENTARII OIM 2012
MAR DEL PLATA – ARGENTINA

Abstract. Comments and solutions for IMO 2012.

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1. Introducere

Prezentarea soluţiilor problemelor propuse la OIM 2012 Mar del Plata (Argentina), augmentate cu comentarii. Multe dintre soluţiile care urmează sunt culese de pe `mathlinks.ro`, cu clarificările şi adăugirile de rigoare (în măsura posibilităţilor – nu garantez întotdeauna acurateţea afirmaţiilor din unele variante).

Din acest motiv, prezentarea de faţă este păstrată în **limba Engleză.**

2. Day I

Problem 1. *Given triangle ABC, the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.*

(The **excircle** *of ABC opposite the vertex A is the circle that is tangent to the side BC, to the ray AB beyond B, and to the ray AC beyond C.)*

Evangelos Psychas - Greece

Solution. It is easy to see that $\angle LFJ = \frac{1}{2}\angle A$ and so the quadrilateral $AFJL$ is cyclic. But $\angle JLA = 90^\circ$ and so $\angle AFJ = 90^\circ$. Thus $AB = BS$, hence $MS = AK$. Similarly $MT = AL$, but $AK = AL$ (as tangents from a same point), and so we are done.

For a slightly different start, let P, Q be the midpoints of MK, ML. Clearly $FP \perp GP$ and $GQ \perp FQ$ so $FPQG$ is cyclic. Thus $\angle FGP = \angle FQP = \angle MQP = \angle MLK$ ($PQ \parallel KL$ of course). Thus $FGLK$ is cyclic. Angle-chasing shows that this angle is actually $\frac{1}{2}\angle B$, which is equal to $\angle FJK$ and so J also lies on this circle. But A is clearly the diameter of (JKL), so the points A, F, K, J, L, G are concyclic. □

Alternative Solutions. Notice that MK and ML are parallel to the internal bisectors of B and C respectively. Then $MK \perp BJ$, $ML \perp CJ$, so M is the orthocentre of JFG. Since $JM \perp BC$ from the tangency, and $JM \perp FG$ from the orthocentre, it follows $BC \parallel FG$. Now we are in a position to prove that $MF \parallel AG$ and $MG \parallel AF$ with any of the methods above, or by considering the excentres I_b, I_c, the fact that I_bI_c is antiparallel to BC, and a short angle chase. With this, FMG is the medial triangle of AST and we are done.

We angle chase on $\triangle FBM$ and easily get $\angle BFL = \frac{1}{2}\angle A$, so $AFCJ$ is cyclic because $\angle LAJ = \frac{1}{2}\angle A$. Then $\angle AFJ = 180^\circ - \angle ALJ = 90^\circ$, so $AS \perp FJ \perp KM$, and $AS \parallel MG$. Similarly $FM \parallel AT$. Since $FBJ \perp AFS$, by symmetry F is the midpoint of AS, and similarly G is the midpoint of AT, so FGM is the medial triangle of $\triangle AST$ and M is the midpoint of ST.

You can actually prove that $AMLT$ and $AMKS$ are isosceles trapezia, so $SM = AK$, $MT = AL$, $AK = AL$ so done. Use simple trigonometry to prove it. □

Alternative Solutions. (Computational Considerations)

[Barycentric Coordinates] Use as reference $\triangle ABC$. It is obvious that $K = (-(s-c) : s : 0)$, $M = (0 : s-b : s-c)$. Also, $J = (-a : b : c)$. In no time one gets

$$G = \left(-a : b : \frac{-as + (s-c)b}{s-b}\right).$$

It follows immediately that

$$T = \left(0 : b : \frac{-as + (s-c)}{s-b}\right) = (0 : b(s-b) : b(s-c) - as).$$

Normalizing, we see that $T = \left(0, -\frac{b}{a}, 1 + \frac{b}{a}\right)$, from which we quickly obtain $MT = s$. Similarly, $MS = s$, and we are done.

[Complex Numbers] Let the excircle be the unit circle and $m = 1$. Then $s = \frac{2k}{k+l}$ and $t = \frac{2l}{k+l}$. □

Alternative Solution. (Darij Grinberg) Guess I'm hardly breaking any news here, but the problem is pretty close to known facts.

First, replace "excircle opposite the vertex A" by "incircle" throughout the problem. This doesn't change the validity of the problem (a phenomenon called "extraversion", and somewhat subtle in the cases when several excircles are concerned; but in our case it's very obvious: just rewrite the problem in terms of triangle KLM instead of triangle ABC, and use the fact that an algebraic-identity type problem that holds for any obtuse-angled triangle

(with obtuse angle at a specified vertex) must also hold for any acute-angled triangle).

"An Unlikely Concurrence" (Chapter 3, 4, page 31 in Ross Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*) now states that the lines BJ, ML and the perpendicular from A to BJ are concurrent. (I have switched A and B.) In other words, $AF \perp BJ$. As a consequence, T is the reflection of A in BJ (because $AF \perp BJ$ and $\angle ABF = \angle FBT$). Combined with the fact that K is the reflection of M in BJ, we see that $AK = SM$ (since reflections leave lengths invariant). Similarly, $AL = TM$. Thus, $SM = AK = AL = TM$ (where $AK = AL$ is for obvious reasons). This is not an equality of directed lengths, but it is easy to see (by reflection again) that S and T lie on different sides of M along the line BC, so we get $SM = MT$ as an equality of directed lengths, and thus M is the midpoint of ST. □

Remarks. (Alexander Magazinov) I have a strong negative opinion on this problem. I dislike the problems where simultaneously a) simple objects are constructed quite complicatedly, b) the question is not to determine this simple object, but to prove some property of it. In this case b) is really annoying.

Problem 2. *Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that*

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Angelo di Pasquale - Australia

Solution. Notice that $1 + a_k = \underbrace{\frac{1}{k-1} + \cdots + \frac{1}{k-1}}_{k-1 \text{ times}} + a_k \geq k \sqrt[k]{\frac{a_k}{(k-1)^{k-1}}}$.

Therefore $(1+a_k)^k \geq \frac{k^k}{(k-1)^{k-1}} a_k$. Taking the product from $k = 2$ to $k = n$ we see that it telescopes to

$$\prod_{k=2}^{n} (1+a_k)^k \geq n^n a_2 a_3 \cdots a_n = n^n.$$

Equality holds if and only if $a_k = \frac{1}{k-1}$ for all $2 \leq k \leq n$, which is not possible since then $\prod_{k=2}^{n} a_k = \frac{1}{(n-1)!} \neq 1$. □

Alternative Solutions. Actually, if we consider the functions $f_k\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ given by $f_k(x) = \dfrac{(1+x)^k}{x}$, then $f_k'(x) = \dfrac{(1+x)^{k-1}((k-1)x-1)}{x^2}$, so f_k reaches its minimum at $x_k = \dfrac{1}{k-1}$. This proves everything, since $f_k(x_k) = \dfrac{k^k}{(k-1)^{k-1}}$ yields (by telescoping) that LHS is at least $n^n a_2 a_3 \cdots a_n = n^n$, with equality if and only if $a_k = \dfrac{1}{k-1}$ for each k, when it is impossible to have $a_2 a_3 \cdots a_n = 1$.

(Dan Schwarz) This result allows a proof by induction. Denote $E(n) = \prod_{k=2}^{n}(1+a_k)^k$ and assume as induction hypothesis that $E(n) \geq n^n \prod_{k=2}^{n} a_k$. The starting case is trivial for $n = 2$, and $E(n+1) = E(n)(1+a_{n+1})^{n+1} \geq \left(n^n \prod_{k=2}^{n+1} a_k\right) \dfrac{(1+a_{n+1})^{n+1}}{a_{n+1}} \geq \left(n^n \prod_{k=2}^{n+1} a_k\right) \dfrac{(n+1)^{n+1}}{n^n} = (n+1)^{n+1} \prod_{k=2}^{n+1} a_k$, while equality cannot occur, since all equality cases are not simultaneously compatible. □

Alternative Solution. For $k \geq 2$

$$(a_k+1)^k \geq \frac{k^k}{(k-1)^{k-1}} a_k$$

is equivalent to

$$\left(\frac{k-1}{k}(a_k+1)\right)^k \geq k\left(\frac{k-1}{k}(a_k+1) - 1\right) + 1,$$

which is true from the well known inequality $x^k \geq k(x-1)+1$ for $x \geq 0$, which is just the Bernoully inequality $(1+(x-1))^k \geq 1+k(x-1)$. Since in our case the exponent k is a positive integer, it can be proved by simple induction. For $k = 1$ it is true, being an identity; and $(1+(x-1))^{k+1} = (1+(x-1))^k(1+(x-1)) \geq (1+k(x-1))(1+(x-1)) = k(x-1)^2 + 1 + (k+1)(x-1) \geq 1+(k+1)(x-1)$.

As it can be seen, the problem is related to the tangent line. □

Remarks. (Dan Schwarz) For the least eligible value $n = 3$ it is elementary to find the true minimum, which is obtained at $(a_1, a_2) = (3/2, 2/3)$ and is equal to $\dfrac{3125}{108} \approx 28.93 > 27 = 3^3$. For $n = 4$ WolframAlpha offers the minimum to be $\approx 359.68 > 256 = 4^4$. It would be interesting to know the true asymptotic for the minimum, and I have some hope Ilya Bogdanov will provide one in his official solution and comments.

Remarks (Sequel)**.** Equivalent to find minimum of $\sum (i \ln (1 + e^{x_i}))$ subject to $x_2 + \cdots + x_n = 0$. Using Lagrange multipliers on the function

$$\sum (i \ln (1 + e^{x_i})) - \lambda(x_2 + ... + x_n)$$

it is easy to see that the minimum occurs when $i\dfrac{e^{x_i}}{1 + e^{x_1}} = \lambda$ for each $2 \leq i \leq n$.

This implies the minimum value occurs when $a_i = \dfrac{\lambda}{i - \lambda}$. It's not hard to check that there's a unique λ (via monotonicity) which makes this all satisfy $\prod a_i = 1$, and the boundary cases are all trivial (unless I've missed something). Anyways this λ satisfies

$$\lambda^{n-1} = \prod (i - \lambda)$$

and we wish to minimize

$$\frac{\prod (i^i)}{\prod \left((i - \lambda)^i\right)}$$

but I can't think of any way to bound either λ or the product in the denominator.

Here is another way of seeing that λ must be very close to 2 as n becomes large.

First, if in $i\dfrac{e^{x_i}}{1 + e^{x_i}} = \lambda$ we set $i = 2$, then we derive $\lambda < 2$.

Secondly, with $\lambda < 2$ the right hand side of

$$\lambda^{n-1} = \prod (i - \lambda)$$

is at least $(2 - \lambda) \cdot 1 \cdot 2 \cdot 3 \cdots (n - 2)$. The left hand side is at most 2^{n-1}. This implies $(2 - \lambda) < 2^{n-1}/(n - 2)!$. Since $(n - 2)!$ grows much much faster than 2^{n-1}, we conclude that $2 - \lambda$ must be very close to 0.

(Alexander Magazinov) As before, under the assumption that $\dfrac{a_i}{1 + a_i} = \dfrac{\lambda}{i}$ we have $A = (1 + a_2)^2 \cdots (1 + a_n)^n = \dfrac{2^2 3^3 \cdots n^n}{(2 - \lambda)^2 (3 - \lambda)^3 \cdots (n - \lambda)^n}$

Since $(2 - \lambda) \cdots (n - \lambda) = \lambda^{n-1}$, we have

$$A = n^n (n - 1)^{n-1} \left(\frac{n - 2}{n - \lambda}\right)^{n-2} \cdots \frac{1}{3 - \lambda} \cdot \lambda^{-2(n-1)}.$$

Further, by Bernoulli, $\left(\dfrac{k - 2}{k - \lambda}\right)^{k-2} \geq 1 - \dfrac{(k - 2)(2 - \lambda)}{k - \lambda} \geq \lambda - 1.$

So, $A \geq n^n \Big((n - 1)^{n-1} \lambda^{-2(n-1)} (\lambda - 1)^{n-2}\Big)$. Or, one can try a more compact, slightly rougher estimate, namely $A \geq n^n (n - 1)^{n-1} \left(\dfrac{\lambda - 1}{\lambda^2}\right)^{n-1}$.

Problem 3. *The* **liar's guessing game** *is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.*

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with **YES** *or* **NO**, *but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.*

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that

1. If $n \geq 2^k$, then B can guarantee a win.

2. For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

DAVID ARTHUR - CANADA

Solution. For just part 1 (although this probably helps part 2).

First, we notice that it obviously doesn't matter what the actual elements that are being guessed are. So we'll generalize the game such that A chooses a finite set D, tells the entire set to B, and picks $x \in D$ for B to guess; clearly this game is still equivalent to the original one.

LEMMA. With a fixed k and n, player B can guarantee a win for all N if and only if B can guarantee a win for $N = n+1$.

Proof. The "only if" part is trivial. For the "if" part, let us use induction on $N \geq n+1$ (for $N \leq n$ just pick $X = D$). The starting case $N = n+1$ is now given to be winning.

For larger $N > n+1$, arbitrarily partition D into $n+1$ nonempty sets $E_1, E_2, \ldots, E_{n+1}$ (they will act as $n+1$ "molecules" made of initial elements, acting as "atoms").

B can use his strategy for $n+1$ elements on $D' := \{1, 2, \ldots, n+1\}$, replacing each question S' with the set $S = \bigcup_{i \in S'} E_i$. Then his strategy will yield a subset X' of D' that has size at most n, and B will know that $x \in X = \bigcup_{i \in X'} E_i$. Since $X' \subsetneq D'$, we have $X = \bigcup_{i \in X'} E_i \subsetneq \bigcup_{i \in D'} E_i = D$ so $|X| < |D| = N$. From here, B has a winning strategy by the induction hypothesis. ■

Trivially, a strategy for $n = 2^k$ is also a strategy for $n \geq 2^k$, so we only need to consider $n = 2^k$, and by the lemma we only have to consider the case $N = 2^k + 1$. Now all B needs to win is to know one element $d \in D$ that cannot be x.

Identify each element of D with a binary string of length k, with one extra element e left over. Let S_i be the set of elements of D corresponding to binary strings with 0 in the i-th position.

First, B asks the questions $S_1, S_2, \ldots, S_k$. Let $a_i = 0$ if A answers $x \in S_i$, and $a_i = 1$ otherwise. Take the binary digit $\overline{a_i} = (1 - a_i)$. Now, let w_i be the element in D corresponding to the binary word $a_1 a_2 \ldots a_i \overline{a_{i+1} a_{i+2} \ldots a_k}$ for $i = 0, 1, \ldots, k$. B next asks the questions $\{w_0\}, \{w_1\}, \ldots, \{w_k\}$ in order.

If A answers at least once that $x \neq w_i$, let j be the smallest nonnegative integer for which A answers this. Then if $x = w_j$, A will have lied for the $k+1$ consecutive questions $S_{j+1}, S_{j+2}, \ldots, S_k, \{w_0\}, \{w_1\}, \ldots, \{w_j\}$. Therefore B knows that $x \neq w_j$ and he wins.

If A always answers that $x = w_i$, then if $x = e$ it means A will have lied for $k + 1$ consecutive questions $\{w_0\}, \{w_1\}, \ldots, \{w_k\}$. Therefore B knows that $x \neq e$ and he again wins. □

3. Day II

Problem 4. *Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds*

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

Liam Baker - South Africa

Solution. (Dan Schwarz) I will try to put some order and method into solving this problem, since all other solutions I've seen are either flawed, or else sway allover the place.

We immediately get, like everybody else does, $f(0) = 0$, and also the completely equivalent form

$$\big(f(a) + f(b) - f(-a-b)\big)^2 = 4f(a)f(b).$$

For $b = 0$ we get $f(a) = f(-a)$ (so f is an even function). Our relation, liberally used in the sequel, writes now

$$\big(f(a+b) - f(a) - f(b)\big)^2 = 4f(a)f(b).$$

Clearly, if $f(1) = 0$, having $(f(a+1) - f(a) - f(1))^2 = 4f(a)f(1) = 0$ yields $f(a+1) = f(a)$ for all a, hence $\boxed{f \equiv 0}$ (which trivially verifies).

Assume therefore $f(1) \neq 0$. If $\boxed{f(k) = k^2 f(1)}$ for all integers k, this clearly provides a solution, easily verified. Since $f(k) = k^2 f(1)$ is true for $k = 1$, assume then $f(k) = k^2 f(1)$ for all $1 \leq k \leq m$, but not for $k = m+1$. We do have $(f(m+1) - f(m) - f(1))^2 = 4f(m)f(1)$, yielding $(f(m+1) - (m^2+1)f(1))^2 = 4m^2 f(1)^2$, whence $f(m+1) = (m^2+1)f(1) - 2mf(1) =$

$(m-1)^2 f(1)$ (the other possibility, $f(m+1) = (m^2+1)f(1) + 2mf(1) = (m+1)^2 f(1)$, is ruled out by our assumption this formula does not continue for $k = m+1$).

Now, on one hand $(f(2m) - f(m) - f(m))^2 = 4f(m)f(m)$ yields $(f(2m) - 2m^2 f(1))^2 = 4m^4 f(1)^2$, leading to

$$f(2m)(f(2m) - 4m^2 f(1)) = 0,$$

while on the other hand $(f(2m) - f(m+1) - f(m-1))^2 = 4f(m+1)f(m-1)$ yields $(f(2m) - 2(m-1)^2 f(1))^2 = 4(m-1)^4 f(1)^2$, leading to

$$f(2m)(f(2m) - 4(m-1)^2 f(1)) = 0.$$

The only possibility to satisfy both relations is $f(2m) = 0$. This implies $(f(n+2m) - f(n) - f(2m))^2 = 4f(n)f(2m) = 0$, forcing $f(n+2m) = f(n)$ for all $n \in \mathbb{Z}$, thus f is periodic of period length $2m$. Then

$$f(m+k) = f((m+k) - 2m) = f(-(m-k)) = f(m-k) = (m-k)^2 f(1)$$

for all $1 \leq k \leq m$.

This offers a solution, both for $\boxed{m=1}$ (when $f(n) = 0$ for $n \equiv 0 \pmod 2$, and $f(n) = f(1)$ for $n \equiv 1 \pmod 2$) and for $\boxed{m=2}$ (when $f(n) = 0$ for $n \equiv 0 \pmod 4$, $f(n) = f(1)$ for $n \equiv 1, 3 \pmod 4$, and $f(n) = 4f(1)$ for $n \equiv 2 \pmod 4$), as is immediately verified by what just has been proved in the above.

Let us finally prove that for $m > 2$ we get a contradiction. Indeed, then $(f(m+1) - f(m-1) - f(2))^2 = 4f(m-1)f(2)$ yields $(4f(1))^2 = 16(m-1)^2 f(1)^2$, whence $(m-1)^2 = 1$, absurd. □

Problem 5. *Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.*

Josef "Pepa" Tkadlec - Czech Republic

Solution. $AL^2 = AD \cdot AB$; then $\angle ALD = \angle LBA$ and a tangent to circumcircles is obvious.

Let F be the intersection of CD and the perpendicular to AL through L; then $FLDA$ is cyclic and $\angle DFA = \angle DLA = \angle LBA$.

Let T be the intersection of BX with AF; then $BFTD$ is cyclic, and X is orthocentre, with AX perpendicular to BF.

In a similar way, the perpendicular for BK through K passes through F, and with similar triangles we can prove that $LF = KF$; then the circumcircle of LKF is tangent to LM and KM, and so M is the radical center, thus finish. □

Alternative Solutions. MK and ML are both tangents to a circle. Let BX meet the circle (A, AC) at J and AX meet the circle (B, BC) at I. Easily we can find that $JKLI$ is cyclic, and easily we can find that BK is tangent

to that circle (because we have $BC \cdot BC = BI \cdot BJ$, and we have $BC = BK$, so $BK \cdot BK = BL \cdot BJ$). Similarly we can see that AL is tangent to the circle too. So MK and ML are both tangents to the circle, so $MK = ML$.

Obviously the perpendiculars through K, L and to BK, CL intersect at a point E which lies on CD, which is also the intersection of $\odot BDK$ and $\odot CDL$. So, by some angle chasing, it follows that $\angle EKD = \angle AXE$, this meaning that EK is tangent to $\odot KDX$. So, by the power of the point E, it follows that $EK^2 = ED \cdot EX = EL^2$, because EL is also tangent to $\odot LDX$. Therefore $EK = EL$, and $MKE \cong MLE$, leading to $MK = ML$. □

Alternative Solution. [Analytical] (Pavel Kozlov)

Let us introduce carthesian coordinates $D(0,0), A(-a,0), B(b,0), X(0,d)$. So the point C has coordinates $(0, \sqrt{ab})$. Suppose the points L and K have coordinates (x_L, y_L) and (x_K, y_K) respectively. The point L lies on the line BX given by equation $y = d - x\frac{d}{b}$ and on the circle with the center $A(-a,0)$ and radius $|AC| = \sqrt{a^2+ab}$ given by equation $(x+a)^2 + y^2 = a^2 + ab$. Then it's easy to see that x_L is positive root of the quadratic equation $(x+a)^2 + (d - x\frac{d}{b})^2 - a^2 - ab = 0$. Accurate calculation give us the next relation $x_L = b\frac{\sqrt{\Pi b} - (ab-d^2)}{b^2+d^2}$, where $\Pi = (ab-d^2)(a+b)$ is symmetrical with respect to a, b. Hence $y_L = d - x_L\frac{d}{b} = d\frac{b^2+ab-\sqrt{\Pi b}}{b^2+d^2}$. Analogously $x_K = -a\frac{\sqrt{\Pi a}-(ab-d^2)}{a^2+d^2}$, $y_K = d\frac{a^2+ab-\sqrt{\Pi a}}{a^2+d^2}$.

So, we are already have some roots and fractions, and it's quite impractical to involve the point M in our calculations. We exclude it with help of the sine's theorem.

From the triangles AKM and BLM we get $\frac{KM}{AK} = \frac{\sin\angle KAM}{\sin\angle KMA}$ and $\frac{LM}{BL} = \frac{\sin\angle LBK}{\sin\angle LNB}$ therefore $\frac{KM}{LM} = \frac{AK}{BL}\frac{\sin\angle KAL}{\sin\angle LBK}$.

To exlude angles we apply the sine's theorem to the triangles AXL and BXK: $\frac{XL}{AL} = \frac{\sin\angle KAL}{\sin\angle AXB}$ and $\frac{XK}{BK} = \frac{\sin\angle LBK}{\sin\angle BXA}$ therefore $\frac{\sin\angle KAL}{\sin\angle LBK} = \frac{XL}{XK}\frac{BK}{AL}$.

Combining the last identities of last two subparagraphs we get $\frac{KM}{LM} = \frac{AK}{BL}\frac{XL}{XK}\frac{BK}{AL} = \frac{AK}{XK}\frac{XL}{BL}\frac{BC}{AC}$.

Let's calculate the first fraction: $\frac{AX}{XK} = \frac{x_K+a}{-x_K} = \frac{a^2+ab-\sqrt{(ab-d^2)((a^2+ab)}}{\sqrt{(ab-d^2)(a^2+ab)}-(ab-d^2)} = \frac{\sqrt{a^2+ab}\,(\sqrt{a^2+ab}-\sqrt{ab-d^2})}{\sqrt{ab-d^2}\,(\sqrt{a^2+ab}-\sqrt{ab-d^2})} = \sqrt{\frac{a^2+ab}{ab-d^2}}$. In the same way we conclude $\frac{XL}{BL} = \sqrt{\frac{ab-d^2}{b^2+ab}}$. Taken evident relation $\frac{BC}{AC} = \sqrt{\frac{b^2+ab}{a^2+ab}}$ into account we finally get $\frac{KM}{LM} = \sqrt{\frac{a^2+ab}{ab-d^2}}\sqrt{\frac{ab-d^2}{b^2+ab}}\sqrt{\frac{b^2+ab}{a^2+ab}} = 1$.

P.S. This solution lets deduce that the foot of the angle bisector of $\angle ACB$ lies on the line XM. □

Remarks. Another addition to this problem. Prove the angle bisector of $\angle ACB$ and the lines XM and AB meet each other at one point.

Problem 6. *Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that*

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

DUSAN DJUKIC - SERBIA

Solution. We do need $\sum_{k=1}^{n} \frac{k}{3^{a_k}} \equiv \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \equiv 1 \pmod 2$, whence-forth $\boxed{n \equiv 1 \text{ or } 2 \pmod 4}$, and we will show all of them are in fact eligible, by strong induction. Before the inductive part though, we shall introduce two types of "substitution", as follows

S1 : $\frac{1}{2^{a_k}} = \frac{1}{2^{a_k+1}} + \frac{1}{2^{a_k+1}}$

and $\frac{k}{3^{a_k}} = \frac{k}{3^{a_k+1}} + \frac{2k}{3^{a_k+1}}$.

S2 : $\frac{1}{2^{a_k}} = \frac{1}{2^{a_k+2}} + \frac{1}{2^{a_k+3}} + \frac{1}{2^{a_k+3}} + \frac{1}{2^{a_k+3}} + \frac{1}{2^{a_k+3}} + \frac{1}{2^{a_k+3}} + \frac{1}{2^{a_k+3}}$

and $\frac{k}{3^{a_k}} = \frac{k}{3^{a_k+2}} + \frac{4k-5}{3^{a_k+3}} + \frac{4k-3}{3^{a_k+3}} + \frac{4k-1}{3^{a_k+3}} + \frac{4k+1}{3^{a_k+3}} + \frac{4k+3}{3^{a_k+3}} + \frac{4k+5}{3^{a_k+3}}$.

Starting from the base cases $n = 1, 5, 9$, which can be easily found

$n = 1$: $\frac{1}{1} = \frac{1}{1} = 1$

$n = 5$: $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{9} + \frac{2}{9} + \frac{3}{9} + \frac{4}{27} + \frac{5}{27} = 1$

$n = 9$: $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} =$
$\frac{1}{9} + \frac{2}{27} + \frac{3}{27} + \frac{4}{27} + \frac{5}{27} + \frac{6}{81} + \frac{7}{81} + \frac{8}{81} + \frac{9}{81} = 1$

we will show that

a) $n = 4m + 1 \mapsto n = 4m + 2$. Simply do the first substitution once (by taking $k = 2m + 1$).

b) $n = 4m + 2 \mapsto n = 4(m + 3) + 1$. First we do the second substitution (by taking $k = m + 2$); after that we apply the first substitution repeatedly (by taking $k = 2m + 2, 2m + 3, 2m + 4, 2m + 5, 2m + 6$).

The proof is complete by induction hypothesis. □

4. ÎNCHEIERE

Problema 1 este o uşoară problemă de geometrie sintetică – nimic rău cu asta! Problema 2 este o inegalitate cu o margine extrem de proastă ca acurateţe, şi care sucumbă imediat la "trucul" cu inegalitatea ponderată a mediilor, sau cu metode (mai mult sau mai puţin) analitice, legate de funcţia $f(x) = \frac{(1+x)^k}{x}$. Era mai potrivită ca problemă 4. Problema 4 conţine o analiză delicată (dar fundamental simplă), unde dificultatea este în eleganţa şi claritatea unei expuneri unde niciun caz nu este pierdut, iar toate cazurile parazite sunt eliminate. Era mai potrivită ca problemă 2. Problema 5 este o problemă de geometrie sintetică mai dificilă, care a produs însă mai multe accidente decât se putea prevedea (în plus, există şi soluţii analitice, ceea ce o cam strică puţin).

Problema 6 nu îşi merită locul. Nu o consider o problemă de Teoria Numerelor, ci de elementară Aritmetică, şi orice soluţie va recurge la găsirea unor identităţi (plicticoase) implicând puteri negative ale lui 2 şi 3. Un exerciţiu fără însemnătate (comparaţi de exemplu cu implicaţiile pline de miez ale teoremei Zeckendorf, relativă la reprezentări ca sume de numere Fibonacci).

Problema 3 este problema Olimpiadei! probabil merită un articol separat pentru ea însăşi (a fost de altfel aleasă de Terence Tao pentru discuţie în proiectul PolyMath). De aceea nu am insistat prea mult, şi nici măcar nu am atins punctul 2), unde sunt multe întrebări legate de cea mai bună margine care poate fi demonstrată.