Clasa a X-a - Etapa 4 - Problema 3

Enunţ. *Demonstraţi că mulţimile*

$$\mathcal{A}=\left\{f:\mathbb{R}\to\mathbb{R}\left|f\left(\frac{x+2y}{3}\right)=\frac{f\left(x\right)+2f\left(y\right)}{3},\ \forall x,y\in\mathbb{R}\right.\right\}$$

şi

$$\mathcal{B}=\left\{f:\mathbb{R}\to\mathbb{R}\left|f\left(\frac{x+4y}{5}\right)=\frac{f\left(x\right)+4f\left(y\right)}{5},\ \forall x,y\in\mathbb{R}\right.\right\}$$

sunt egale.

***

Soluţie. Fie mulţimea $\mathcal{C}=\left\{f:\mathbb{R}\to\mathbb{R}\left|f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2},\ \forall x,y\in\mathbb{R}\right.\right\}$. Vom demonstra egalitatea $\mathcal{A}=\mathcal{C}$ folosind dubla incluziune.

Fie $f\in\mathcal{C}$. Atunci $f\left(\frac{x+y+z+t}{4}\right)=\frac{f(x)+f(y)+f(z)+f(t)}{4}$, pentru orice $x,y,z,t\in\mathbb{R}$. Alegem $t=\frac{x+y+z}{3}$ şi obţinem $f\left(\frac{x+y+z}{3}\right)=\frac{f(x)+f(y)+f(z)}{3}$. Apoi $z=y$, conduce la $f\in\mathcal{A}$.

Reciproc, fie $f\in\mathcal{A}$. Folosim identitatea de le punctul anterior în cazul $a=\frac{2}{3}$ şi obţinem

$$\begin{aligned}
f\left(\frac{x+y}{2}\right)&=f\left(\frac{2}{3}\left(\frac{x}{3}+\frac{2}{3}\cdot\frac{x+y}{2}\right)+\frac{1}{3}\left(\frac{2}{3}y+\frac{1}{3}\cdot\frac{x+y}{2}\right)\right)\\
&=\frac{2}{3}f\left(\frac{x}{3}+\frac{2}{3}\cdot\frac{x+y}{2}\right)+\frac{1}{3}f\left(\frac{2}{3}y+\frac{1}{3}\cdot\frac{x+y}{2}\right)\\
&=\frac{2}{3}\left(\frac{1}{3}f\left(x\right)+\frac{2}{3}f\left(\frac{x+y}{2}\right)\right)+\frac{1}{3}\left(\frac{2}{3}f\left(y\right)+\frac{1}{3}f\left(\frac{x+y}{2}\right)\right)\\
&=\frac{2}{9}\left(f\left(x\right)+f\left(y\right)\right)+\frac{5}{9}f\left(\frac{x+y}{2}\right).
\end{aligned}$$

de unde obţinem $f\in\mathcal{C}$, adică $\mathcal{A}=\mathcal{C}$.

Similar demonstrăm egalitatea $\mathcal{B}=\mathcal{C}$, de unde obţinem concluzia. $\square$